The Gaussian free field in interlacing particle systems

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Abstract

We show that if an interlacing particle system in a two-dimensional lattice is a determinantal point process, and the correlation kernel can be expressed as a double integral with certain technical assumptions, then the moments of the fluctuations of the height function converge to that of the Gaussian free field. In particular, we examine a specific particle system that was previously studied in [3] and compute the corresponding Green's function.

1 Introduction

The Gaussian free field is widely considered to be a universal object describing the flucutations of heights of random surfaces. Previous work has rigorously shown this to be the case in specific models ([2],[10]). In this paper, we show that if an interlacing particle system in two dimensions can be described as a determinantal point process, and mild assumptions are made about the correlation kernel, then the covariances of the fluctuations of the height function are governed by a particular Green's function. A general formula for the Green's function is given.

In particular, we will use this general theorem to show that in an interlacing particle system that arises from the representation theory of the orthogonal groups, the Green's function is

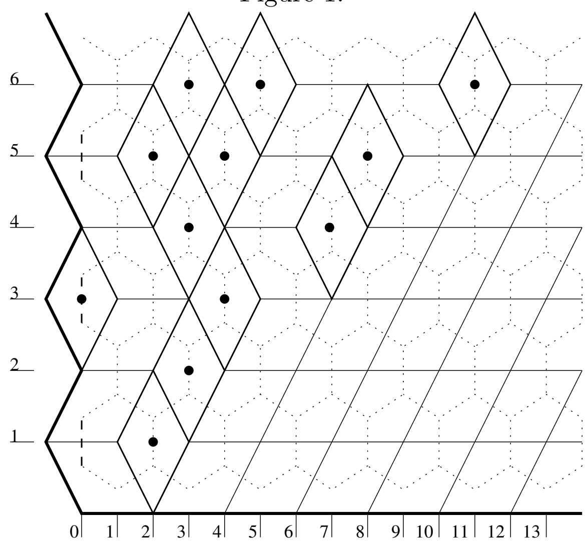
$$\mathcal{G}(z,w)=\frac{1}{2\pi}\log\left(\frac{z+z^{-1}-\bar{w}-\bar{w}^{-1}}{z+z^{-1}-w-w^{-1}}\right).$$

Note that $\mathcal{G}$ is the Green's function for the Laplace operator with Dirichlet boundary conditions on the set $\mathbb{H}-\mathbb{D}:=\{z\in\mathbb{C}:\Im z>0 \text{ and } |z|>1\}$. It turns out that there is a map Ω from the surface to $\mathbb{H}-\mathbb{D}$. We will show that the fluctuations of the height function converge to a Gaussian process whose covariance is given by the pullback of $\mathcal{G}$ under Ω.

Particle System. Now let us describe this particle system, which was the initial motivation for this paper. Introduce coordinates on the plane as shown

in Figure 1. Denote the horizontal coordinates of all particles with vertical coordinate m by $y_1^m > y_2^m > \cdots > y_k^m$, where $k = \lfloor (m+1)/2 \rfloor$. There is a wall on the left side, which forces $y_k^m \geq 0$ for m odd and $y_k^m \geq 1$ for m even. The particles must also satisfy the interlacing conditions $y_{k+1}^{m+1} < y_k^m < y_k^{m+1}$ for all meaningful values of k and m.

Figure 1:



By visually observing Figure 1, one can see that the particle system can be interpreted as a stepped surface. We thus define the height function at a point to be the number of particles to the right of that point.

Define a continuous time Markov chain as follows. The initial condition is a single particle configuration where all the particles are as much to the left as possible, i.e. $y_k^m = m - 2k + 1$ for all k, m. This is illustrated in the left-most iamge in Figure 2. Now let us describe the evolution. We say that a particle y_k^m is blocked on the right if $y_k^m + 1 = y_{k-1}^{m-1}$, and it is blocked on the left if $y_k^m - 1 = y_k^{m-1}$ (if the corresponding particle y_{k-1}^{m-1} or y_k^{m-1} does not exist, then y_k^m is not blocked).

Each particle has two exponential clocks of rate $\frac{1}{2}$; all clocks are independent. One clock is responsible for the right jumps, while the other is responsible for the left jumps. When the clock rings, the particle tries to jump by 1 in the corresponding direction. If the particle is blocked, then it stays still. If the particle is against the wall (i.e. $y^m_{[\frac{m+1}{2}]} = 0$) and the left jump clock rings, the particle is reflected, and it tries to jump to the right instead.

When y_k^m tries to jump to the right (and is not blocked on the right), we find the largest $r \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ such that $y_k^{m+i} = y_k^m + i$ for $0 \leq i \leq r$, and the jump consists of all particles $\left\{y_k^{m+i}\right\}_{i=0}^{r}$ moving to the right by 1. Similarly,

when y_k^m tries to jump to the left (and is not blocked on the left), we find the largest $l \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ such that $y_{k+j}^{m+j} = y_k^m - j$ for $0 \leq j \leq l$, and the jump consists of all particles $\left\{y_{k+j}^{m+j}\right\}_{j=0}^{l}$ moving to the left by 1.

In other words, the particles with smaller upper indices can be thought of as heavier than those with larger upper indices, and the heavier particles block and push the lighter ones so that the interlacing conditions are preserved.

Figure 2: First three jumps

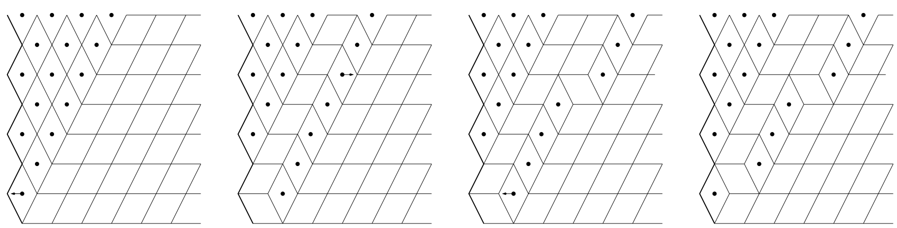


Figure 2 depicts three possible first jumps: Left clock of y_1^1 rings first (it gets reflected by the wall), then right clock of y_1^5 rings, and then left clock of y_1^1 again.

In terms of the underlying stepped surface, the evolution can be described by saying that we add possible "sticks" with base 1×1 and arbitrary length of a fixed orientation with rate $1/2$, remove possible "sticks" with base 1×1 and a different orientation with rate $1/2$, and the rate of removing sticks that touch the left border is doubled.[1]

A computer simulation of this dynamics can be found at
`http://www.math.caltech.edu/papers/Orth_Planch.html`.

This particle system falls in the universality class of the Anisotropic Kardar-Parisi-Zhang (AKPZ) equation with a wall. The KPZ equation was first introduced in [8] and is of interest to physicists, see [5]. Similar Markov chains have been previously studied in [2] without the wall, and in [13] with a different ("symplectic") interaction with the wall.

More general particle systems. By following the proof, the author realized that a more general statement could be proved. If a point process in a two-dimensional lattice is determinantal, and the correlation kernel can be expressed as a double integral with certain technical assumptions (see Definition 2.1 below), then the moments of the fluctuations of the height function can be governed by a Green's function. The exact statement is Theorem 2.3. We then

[1]This phrase is based on the convention that is a figure of a $1 \times 1 \times 1$ cube. If one uses the dual convention that this is a cube-shaped hole then the orientations of the sticks to be added and removed have to be interchanged, and the tiling representations of the sticks change as well.

can use this theorem to determine the Green's function for the specific point process described above.

Motivations. There are three reasons for proving the results in the more general case. The first reason is that the proofs are not much more difficult. The second reason is that it is easier to check the conditions for the general case than it is to repeat the full calculations. The third reason is that the general result tells us that the formula for the Green's function only depends on G_ν, and therefore should only depend on the horizontal movement of the particles. For instance, in the model of [2], the expression for $\mathcal{G}$ was explicitly computed. In the model of [6], the particles had two different jump rates, and the expression for $\mathcal{G}$ was the same. Our general results should imply, for example, that $\mathcal{G}$ will be the same for any number of different jump rates. In order for a model to have a different expression for $\mathcal{G}$, there should be some change along the horizontal direction. For example, in the model considered in this paper, a reflecting wall is added to the model of [2], and we see that $\mathcal{G}$ is changed.

The correlation kernels in previous problems ([4],[6]) should satisfy the conditions of the more general case, although the author has not rigorously checked the details.

Conjectures. There are several statements which should be true, but are not pursued in this paper. One of them concerns the one-point fluctuations. Namely, after a scaling of $\sqrt{\log N}$, the fluctuations of the height function at a single point should converge in moments to a standard Gaussian. The proof would consist mainly in proving that the variance is proportional to $\log N$, which by [12] immediately implies the convergence to a Gaussian.

The appropriate scaling was first demonstrated in [14] using a renormalization group argument. Using this convergence, one can then show that the convergence to the Gaussian free field occurs in distribution, not just in moments. Analogous (rigorous) statements are found as Theorem 1.2 and Proposition 5.6 in [2].

It should also be possible to prove similar results when $\tau_1 \neq \tau_2$, but this was also not pursued.

Organization of Paper. In section 2.1, the theorem for more general particle systems (Theorem 2.3) is stated. In sections 2.2 and 2.3, the algebraic and analytic steps of the proof are given. In section 3, it is shown that the particle system described above satisfies the conditions of Theorem 2.3. Section 4.2 deals with the necessary asymptotic analysis.

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2 General Results

2.1 Statement of the Main Theorem

Suppose we have a family of point processes on $\mathfrak{X} = \mathbb{Z}\times\mathbb{Z}_{\geq 1}$ which runs over time $t \in [0,\infty)$. (Note that these are different co-ordinates from the introduction). In other words, at any time t, the system selects a random subset $X \subset \mathfrak{X}$. If $(x,n) \in X$, then we say that there is a particle at (x,n). For any $k \geq 1$ and $t \geq 0$, let $\rho_k^t : \mathfrak{X}^k \to [0,1]$ be defined by

$$\begin{aligned}&\rho_k^t(x_1,n_1,\dots,x_k,n_k)\\ &\qquad = \mathbb{P}(\text{There is a particle at } (x_j,n_j) \text{ at time } t \text{ for each } j=1,\dots,k).\end{aligned}$$

Assume that there is a map K on $\mathfrak{X}\times\mathfrak{X}\times[0,\infty)$ such that

$$\rho_k^t(x_1,n_1,\dots,x_k,n_k) = \det[K(x_i,n_i,x_j,n_j,t)]_{1\leq i,j\leq k}. \tag{1}$$

The maps ρ_k and K are called the *kth correlation function* and the *correlation kernel*, respectively.

A function c on $\mathfrak{X}\times\mathfrak{X}$ is called a *conjugating factor* if there exists another function $\mathcal{C}$ on $\mathfrak{X}$ such that

$$c(x,n,x',n') = \frac{\mathcal{C}(x,n)}{\mathcal{C}(x',n')}.$$

Note that if c is a conjugating factor, then

$$\det[K(x_i,n_i,x_j,n_j,t)]_{1\leq i,j\leq k} = \det[c(x_i,n_i,x_j,n_j)K(x_i,n_i,x_j,n_j,t)]_{1\leq i,j\leq k}. \tag{2}$$

Two kernels K and $\tilde{K}$ are called *conjugate* if $\tilde{K} = cK$ for some conjugating factor c.

If a correlation kernel exists, the point process is called *determinantal*. On a discrete space, a point process is determined uniquely by its correlation functions (see e.g. [9]). Therefore, if we are given two determinantal point process on a discrete space with conjugate kernels, they must have the same law.

The set $\mathbb{Z}\times\{n\}$ is called the *nth level*. Given a subset $X \subset \mathfrak{X}$, let m_n be the cardinality of the set $X \cap (\mathbb{Z}\times\{n\})$. Assume that the numbers m_n take constant finite values which are independent of the time parameter t. In words, this means that the number of particles on the nth level is always m_n. Further assume that $m_n \leq m_{n+1} \leq m_n + 1$ for all n. Let $x_1^{(n)} > x_2^{(n)} > \ldots > x_{m_n}^{(n)}$ denote the elements of $X \cap (\mathbb{Z}\times\{n\})$. A subset X is called *interlacing* if

$$x_{k+1}^{(n+1)} < x_k^{(n)} \leq x_k^{(n+1)}, \quad \text{when} \quad m_{n+1} = m_n,$$

$$x_{k+1}^{(n+1)} \leq x_k^{(n)} < x_k^{(n+1)}, \quad \text{when} \quad m_{n+1} = m_n + 1.$$

Assume that at any time t, the system almost surely selects an interlacing subset. Let δ_n equal $m_{n+1} - m_n$.

Define the random *height function* by

$$h : \mathfrak{X} \times \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0},$$

$$h(x,n,t) = |\{(s,n) \in X : s > x\}|.$$

In words, h counts the number of particles to the right of (x,n) at time t.

We wish to study the large-time asymptotics of this particle system. Let $x = [N\nu], n = [N\eta], t = N\tau$, where N is a large parameter. Define $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ to be

$$\mathcal{D} := \{(\nu,\eta,\tau) : \lim_{N\to\infty} \rho_1^t(x,n) > 0\}.$$

Let H_N be defined by

$$H_N : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},$$

$$H_N(\nu,\eta,\tau) := h(x,n,t) - \mathbb{E}h(x,n,t).$$

In words, H_N is the fluctuation of the height function around its expectation.

Before stating the theorem, we need to state some more assumptions on the kernel.

Suppose the kernel K is conjugate to a kernel $\tilde{K}$ such that $\tilde{K}$ satisfies the following property: There is a number L such that whenever $x, x' \geq L$,

$$\tilde{K}(x,n,x',n',t) + \tilde{K}(x,n,x'-1+\delta_n,n'+1,t) + \tilde{K}(x,n,x'+\delta_n,n'+1,t) = \begin{cases} 1, & (x,n) = (x',n') \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

$$\tilde{K}(x,n,x',n',t) + \tilde{K}(x+1-\delta_n,n-1,x',n',t) + \tilde{K}(x-\delta_n,n-1,x',n',t) = \begin{cases} 1, & (x,n) = (x',n') \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Further suppose that for $x', x'' > L$,

$$\tilde{K}(x,n,x',n',t)\tilde{K}(x'',n'',x-1+\delta_n,n+1,t) \to 0 \text{ as } x \to \infty \tag{5}$$

$$\tilde{K}(x,n,x',n',t)\tilde{K}(x'',n'',x+\delta_n,n+1,t) \to 0 \text{ as } x \to \infty \tag{6}$$

$$\tilde{K}(x,n,x-1+\delta_n,n+1,t) \to 0 \text{ as } x \to \infty \tag{7}$$

$$\tilde{K}(x,n,x+\delta_n,n+1,t) \to 1 \text{ as } x \to \infty. \tag{8}$$

Suppose $G(\nu,\eta,\tau,z)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{C}$. To save space, we will sometimes write $G(z)$. Expressions such as G', G_ν, G'_ν will be shorthand for $\partial G/\partial z$, $\partial G/\partial \nu$ and $\partial^2 G/\partial z \partial \nu$, respectively. Assume $G(\overline{z}) = \overline{G(z)}$.Also suppose there exists a differentiable map Ω from $\mathcal{D}$ to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ such that Ω is a critical point of G. In other words,

$$G'(\nu,\eta,\tau,\Omega(\nu,\eta,\tau)) = 0 \text{ for all } (\nu,\eta,\tau) \in \mathcal{D}. \tag{9}$$

Note that Ω need not be onto. For any (η,τ), if the set $\{\nu\in\mathbb{R}:(\nu,\eta,\tau)\in\mathcal{D}\}$ is nonempty, let $q_2(\eta,\tau)$ denote its supremum.

Definition 2.1. With the notation above, a determinantal point process on $\mathbb{Z}\times\mathbb{Z}_{\geq 1}$ is *normal* if all of the following hold:

- For all $\eta,\tau>0$, the limit $\Omega(q_2(\eta,\tau)-0,\eta,\tau)$ exists and is a positive real number.
- For all $\eta,\tau>0$, as ν approaches $q_2(\eta,\tau)$ from the left, $G''(\nu,\eta,\tau,\Omega(\nu,\eta,\tau))=\mathcal{O}((q_2(\eta,\tau)-\nu)^{1/2})$.
- K is conjugate to some $\tilde{K}$ such that (3)-(8) hold for some integer L.
- Set $t=N\tau$, $x_j=[N\nu_j]$ and $n_j=[N\eta_j]$ for $j=1,2$, where $(\nu_j,\eta_j,\tau)\in\mathcal{D}$. Let Ω_j denote $\Omega(\nu_j,\eta_j,\tau)$ and let $G_j(z)$ denote $G(\nu_j,\eta_j,\tau,z)$. For finite integers k_1 and k_2, as $N\to\infty$,

  $$\tilde{K}(x_1,n_1+k_1,x_2,n_2+k_2,t)=$$
  $$\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_1}\int_{\Gamma_2}\frac{\exp(NG(\nu_1,\eta_1,\tau,z))}{\exp(NG(\nu_2,\eta_2,\tau,w))}f_{k_1k_2}(u,w)dwdu+\mathcal{O}(e^{N\kappa}),\quad(10)$$

  where Γ_1 and Γ_2 are steepest descent paths, $\kappa<\Re(G_1(\Omega_1)-G_2(\Omega_2))$, and $f_{k_mk_n}$ are complex-valued meromorphic functions satisfying the identity

  $$\begin{aligned}f_{k_1k_2}(z_1,z_2)f_{k_2k_3}(z_2,z_3)\dots f_{k_{r-1}k_r}(z_{r-1},z_r)f_{k_rk_1}(z_r,z_1)\\ =f(z_1,z_2)f(z_2,z_3)\dots f(z_{r-1},z_r)f(z_r,z_1).\end{aligned}$$

  Here, we have written f for f_{00}.
- For any $l\geq 3$, the following indefinite integral satisfies

  $$\int\cdots\int\sum_{\sigma}\prod_{i=1}^{l}\frac{f(z_{\sigma(i)},z_{\sigma(i+1)})}{G'_\nu(z_{\sigma(i)})}dz_i\equiv 0,\quad(11)$$

  where the sum is taken over all l-cycles in S_l and the indices are taken cyclically.
- For any finite interval $[a,b]$, $G\in C^2[a,b]$ and the Lesbesgue measure of the set $\{x\in[a,b]:I'(x)\in 2\pi\mathbb{Z}+[-\delta,\delta]\}$ is $\mathcal{O}(\delta^a)$ for some positive a.

The following remarks will help explain the definition.

Remark 2.2. (1) The assumption that $\Omega(q_2(\eta,\tau)-0,\eta,\tau)>0$ occurs naturally. One often finds that for $k_1\neq k_2\in\mathbb{Z}$,

$$\lim_{N\to\infty}K([N\nu]+k_1,[N\eta],[N\nu]+k_2,[N\eta],N\tau)=\frac{1}{2\pi i}\int_{\bar{\Omega}}^{\Omega}\frac{dz}{z^{k_1-k_2+1}}=\frac{\Im(\Omega^{k_2-k_1})}{\pi(k_2-k_1)},$$

where the contour crosses the positive real line. By setting $\Omega = e^{i\varphi}$, we see that the right hand side reduces to the ubiquitous sine kernel. When $k_1 = k_2 = 0$, we see that

$$\lim_{N\to\infty} \rho_1^{N\tau}([N\nu],[N\eta]) = \frac{1}{2\pi}(\log\Omega - \log\bar{\Omega}) = \frac{\arg\Omega(\nu,\eta,\tau)}{\pi}.$$

Since the left hand side equals zero, we expect $\arg\Omega(q_2(\eta,\tau)-0,\eta,\tau) = 0$.

(2) Since $G(\overline{z}) = \overline{G(z)}$, this means that G has a critical point at both Ω and $\bar{\Omega}$. As Ω approaches the real line, the two critical points coalesce into a triple zero, so $G''(t,\eta,\tau)$ converges to 0 as t approaches $q_2(\eta,\tau)$. We need a control for how quickly this convergence to 0 occurs, in order to order to control the behavior near the boundary of $\mathcal{D}$. More specifically, it controls the bound in Proposition 4.11.

There is a heuristic understanding for why (2) should hold. The function G has two critical points which coalesce into a triple zero. The simplest example of such a function is $G(t,x) = x^3/3 - tx$ as t approaches 0. In this case, the solution to $G'(t,x) = 0$ is $\Omega(t,x) = t^{1/2}$. Then $G''(t,\Omega(t,x)) = 2t^{1/2}$.

(3) Assumptions (3)–(8) will be elucidated when we interpret the particles as lozenges. In particular, see remark 2.6.

(4) It is common for the kernel to be expressed in this form; previous examples are [4] and [2]. If the kernel has a different expression with the same asymptotics as in Propositions 4.5 and 4.11, the results still hold.

(5) In particular, (11) holds if there always exist u-substitutions and an expression Y such that

$$\int\cdots\int\prod_{i=1}^{l}\frac{f(z_i,z_{i+1})}{G'_{\nu_1}(z_i)}dz_i = \int\cdots\int\prod_{i=1}^{l}\frac{1}{Y(u_i)-Y(u_{i+1})}du_i,$$

where $z_{l+1} = z_1$ and $u_{l+1} = u_1$. This is because of Lemma 7.3 in [7], which refers back to [1], which says that

$$\sum_{\sigma}\prod_{i=1}^{l}\frac{1}{Y(u_{\sigma(i)})-Y(u_{\sigma(i+1)})} = 0.$$

(6) This is a technical lemma which allows Lemma 4.1 to be applied.

We can now state the main theorem.

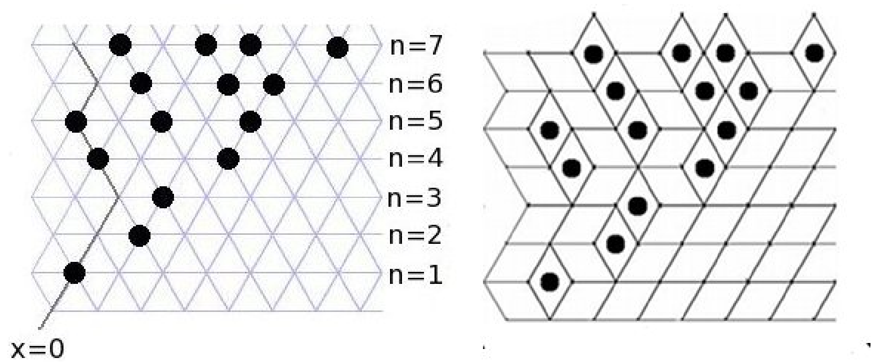
Theorem 2.3. *Suppose we are given a normal determinantal point process. For $1 \le j \le k$, let $\varkappa_j = (\nu_j,\eta_j,\tau)$ be distinct points in $\mathcal{D}$, and let $\Omega_j = \Omega(\nu_j,\eta_j,\tau)$. Define the function $\mathcal{G}$ on the upper half-plane to be*

$$\mathcal{G}(z,w) = \left(\frac{1}{2\pi}\right)^2\int_{\bar{z}}^{z}\int_{\bar{w}}^{w}\frac{f(z_1,z_2)f(z_2,z_1)}{G'_\nu(z_1)G'_\nu(z_2)}dz_2dz_1$$

Then

$$\lim_{N\to\infty}\mathbb{E}(H_N(\varkappa_1)\cdots H_N(\varkappa_k)) = \begin{cases}\displaystyle\sum_{\sigma\in\mathcal{F}_k}\prod_{j=1}^{k/2}\mathcal{G}(\Omega_{\sigma(2j-1)},\Omega_{\sigma(2j)}), & k \text{ is even}\\ 0, & k \text{ is odd,}\end{cases}$$

Figure 3: In this example, the integers m_n equal $1, 1, 1, 2, 3, 3, 4 \ldots$. The black line on the left represents the points where $x = 0$. Examples of $x_k^{(n)}$ are $x_1^{(3)} = 1, x_1^{(4)} = 3, x_2^{(7)} = 4$.



where $\mathcal{F}_k$ is the set of all involutions in S_k without fixed points.

Remark 2.4. We note that these are the moments of a linear family of Gaussian random variables: see Appendix A. Using the results of [12], it should be possible to show that $H_N(\varkappa)/\sqrt{\mathrm{Var} H_N(\varkappa)}$ converges to a Gaussian, but this was not pursued.

2.2 Algebraic steps in proof of Theorem 2.3

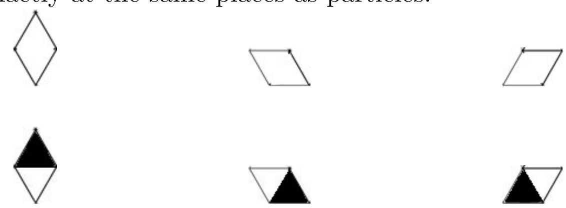
The most natural way to view $\mathfrak{X}$ is as a square lattice. However, it turns out that a hexagonal lattice is more useful. To obtain the hexagonal lattice, take the nth level and shift it to the right by $(n+1)/2 - m_n$. See Figure 3.

Figure 3 also shows that the particle system can be interpreted as lozenges. Each lozenge is a pair of adjacent equilateral triangles. See Figure 4.

By setting the location of each triangle to be the midpoint of its horizontal side, each lozenge can be viewed as a pair (x, n, x', n'), where the black triangle is located at (x, n) and the white triangle is located at (x', n'). For example, in Figure 3 there are lozenges $(1, 3, 1, 3), (2, 3, 2, 4)$, and $(0, 3, 1, 4)$. The three types of lozenges can be described as follows. For lozenges of type I, $(x', n') = (x, n)$. For lozenges of type II, $(x', n') = (x - 1 + \delta_n, n + 1)$. For lozenges of type III, $(x', n') = (x + \delta_n, n + 1)$. Note that a lozenge of type I is just a particle.

We say that $(x, n, x', n') \in \mathfrak{X} \times \mathfrak{X}$ is *viable* if $(x', n') = (x, n), (x - 1 + \delta_n, n + 1)$, or $(x + \delta_n, n + 1)$. A sequence $(x_1, n_1, x_1', n_1'), \ldots, (x_k, n_k, x_k', n_k')$ of viable elements is *non-overlapping* if $(x_1, n_1), \ldots, (x_k, n_k)$ are all distinct from each other and $(x_1', n_1'), \ldots, (x_k', n_k')$ are also all distinct from each other. We do, however, allow the possibility of $(x_i, n_i) = (x_j', n_j')$.

Figure 4: Lozenges of types I,II, and III, respectively. Note that lozenges of type I occur exactly at the same places as particles.



The statement and proof of the next proposition are similar to Theorem 5.1 of kn:BF.

Proposition 2.5. *Suppose the kernel K is conjugate to some $\tilde{K}$ such that* (3)-(8) *hold for some L. If $t \geq 0, x_1, x_1', \ldots, x_k, x_k' > L$, and $(x_1, n_1, x_1', n_1'), \ldots, (x_k, n_k, x_k', n_k')$ is a sequence of non-overlapping viable elements of $\mathfrak{X} \times \mathfrak{X}$, then*

$$\mathbb{P}(\textit{There is a lozenge } (x_j, n_j, x_j', n_j') \textit{ at time } t \textit{ for each } j = 1, \ldots, k) \\ = \det[\tilde{K}(x_i, n_i, x_j', n_j', t)]_{1 \leq i,j \leq k}. \quad (12)$$

Remark 2.6. The equations (3)–(8) can now be intuitively understood. Equation (3) says that each black triangle is located in exactly one of the three lozenges around it, and equation (4) makes an identical statement for white triangles. Equations (5) and (7) say that lozenges of type II almost surely do not occur far to the right of the particles, with (5) controlling the off-diagonal entries in the determinant and (7) controllling the diagonal entries. Similarly, equations (6) and (8) says that lozenges of type III almost surely do occur far to the right of the particles. This intuition will be exploited in the proof of Thereom 2.5.

Proof. We proceed by induction on the number of lozenges that are not of type I. When this number is zero, the statement reduces to (1) and (2).

For any set $S = \{(x_1, n_1, x_1', n_1'), \ldots, (x_k, n_k, x_k', n_k')\}$ of non-overlapping, viable elements, let $P(S)$ and $D(S)$ denote the left and right hand sides of (12), respectively. First, as a preliminary statement, we prove that if $(x_{k+1}, n_{k+1}) \neq (x_r, n_r)$ for $1 \leq r \leq k$, then

$$D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1}, n_{k+1})\}) + D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1} - 1 + \delta_n, n_{k+1} + 1)\}) \\ + D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1} + \delta_n, n_{k+1} + 1)\}) = D(S). \quad (13)$$

Note that if D is replaced by P in (13), the statement is immediate, since the black triangle at (x_{k+1}, n_{k+1}) must be contained in exactly one lozenge.

Expand the determinants into a sum over permutations $\sigma \in S_{k+1}$. Now split the sum into two parts, one part where the σ fixes $k+1$ and one part where σ

does not fix $k+1$. The first part equals

$$\sum_{\sigma(k+1)=k+1} \operatorname{sgn}(\sigma) \prod_{r=1}^{k} \tilde{K}(x_{\sigma(r)}, n_{\sigma(r)}, x'_r, n'_r, t)$$
$$\times \Big[\tilde{K}(x_{k+1}, n_{k+1}, x_{k+1}, n_{k+1}, t) + \tilde{K}(x_{k+1}, n_{k+1}, x_{k+1} - 1 + \delta_{n_{k+1}}, n_{k+1} + 1, t)$$
$$+ \tilde{K}(x_{k+1}, n_{k+1}, x_{k+1} + \delta_{n_{k+1}}, n_{k+1} + 1, t)\Big],$$

which, by (3), equals $D(S)$. The second part equals

$$\sum_{\sigma(k+1)\neq k+1} \operatorname{sgn}(\sigma) \prod_{r=1}^{k} \tilde{K}(x_{\sigma(r)}, n_{\sigma(r)}, x'_r, n'_r, t)$$
$$\times \Big[\tilde{K}(x_{\sigma(k+1)}, n_{\sigma(k+1)}, x_{k+1}, n_{k+1}, t) + \tilde{K}(x_{\sigma(k+1)}, n_{\sigma(k+1)}, x_{k+1} - 1 + \delta_{n_{k+1}}, n_{k+1} + 1, t)$$
$$+ \tilde{K}(x_{\sigma(k+1)}, n_{\sigma(k+1)}, x_{k+1} + \delta_{n_{k+1}}, n_{k+1} + 1, t)\Big].$$

Since $(x_{\sigma(k+1)}, n_{\sigma(k+1)}) \neq (x_{k+1}, n_{k+1})$ by assumption, (3) implies that this expression equals 0. Therefore (13) holds.

In a similar manner, if $(x'_{k+1}, n'_{k+1}) \neq (x'_r, n'_r)$ for $1 \leq r \leq k$, then (4) implies that

$$D(S\cup\{(x_{k+1}, n_{k+1}, x_{k+1}, n_{k+1})\}) + D(S\cup\{(x_{k+1}+1-\delta_n, n_{k+1}-1, x_{k+1}, n_{k+1})\})$$
$$+ D(S \cup \{(x_{k+1} - \delta_n, n_{k+1} - 1, x_{k+1}, n_{k+1})\}) = D(S). \quad (14)$$

Again, the statement holds if D is replaced by P.

Now let us prove that D and P still agree if we add a lozenge of type II to S. Suppose that $(x, n, x-1+\delta_n, n+1)$ is viable and that $S\cup\{(x, n, x-1+\delta_n, n+1)\}$ is non-overlapping. Then equation (13) is equivalent to

$$D(S \cup \{(x, n, x - 1 + \delta_n, n + 1)\})$$
$$= D(S) - D(S \cup \{(x, n, x, n)\}) - D(S \cup \{(x, n, x + \delta_n, n + 1)\}), \quad (15)$$

and the same holds for P instead of D. By the induction hypothesis,

$$D(S) = P(S),$$
$$D(S \cup \{(x, n, x, n)\}) = P(S \cup \{(x, n, x, n)\})$$
$$D(S \cup \{(x + \delta_n, n + 1, x + \delta_n, n + 1)\}) = P(S \cup \{(x + \delta_n, n + 1, x + \delta_n, n + 1)\}).$$

Thus, (15) implies

$$D(S \cup \{(x, n, x - 1 + \delta_n, n + 1)\}) - P(S \cup \{(x, n, x - 1 + \delta_n, n + 1)\}))$$
$$= -D(S \cup \{(x, n, x + \delta_n, n + 1)\}) + P(S \cup \{(x, n, x + \delta_n, n + 1)\}).$$

Assume for now that $(x'_r, n'_r) \neq (x+\delta_n, n+1)$ for $1 \leq r \leq$k. Then we cam apply equation (14), which implies that

$$\begin{aligned} &D(S \cup \{(x, n, x-1+\delta_n, n+1)\}) = D(S) \\ &- D(S \cup \{(x+\delta_n, n+1, x+\delta_n, n+1)\}) - D(S \cup \{(x+1, n, x+\delta_n, n+1)\}), \end{aligned} \tag{16}$$

and the same statement holds for P. Thus,

$$\begin{aligned} &- D(S \cup \{(x, n, x+\delta_n, n+1)\}) + P(S \cup \{(x, n, x+\delta_n, n+1)\}) \\ &\quad = D(S \cup \{(x+1, n, x+\delta_n, n+1)\}) - P(S \cup \{(x+1, n, x+\delta_n, n+1)\}). \end{aligned}$$

If $S \cup \{(x+1, n, x+\delta_n, n+1)\}$ is non-overlappinng, then (15) is again applicable. We repeatedly apply (15) and (16) as often as possible. First, suppose that this can be done indefinitely. Then

$$\begin{aligned} &|D(S \cup \{(x, n, x-1+\delta_n, n+1)\}) - P(S \cup \{(x, n, x-1+\delta_n, n+1)\})| \\ &= \lim_{M\to\infty} |D(S \cup \{(x+M, n, x-1+\delta_n+M, n+1)\}) - P(S \cup \{(x+M, n, x-1+\delta_n+M, n+1)\})|. \end{aligned}$$

Since lozenges of type II almost surely do not appear when we look far to the right of the particles,

$$\lim_{M\to\infty} P(S \cup \{(x+M, n, x-1+\delta_n+M, n+1)\}) = 0.$$

By expanding the determinant into a sum over S_{k+1}, (5) and (7) imply that

$$\lim_{M\to\infty} D(S \cup \{(x+M, n, t, x-1+\delta_n+M, n+1)\}) = 0.$$

Now suppose that (15) and (16) can only be applied finitely many times. This means that $D(S \cup \{(x, n, x-1+\delta_n, n+1)\}) - P(S \cup \{(x, n, x-1+\delta_n, n+1)\})$ equals either

$$D(S \cup \{(x+M, n, x+M+\delta_n, n+1)\}) - P(S \cup \{(x+M, n, x+M+\delta_n, n+1)\})$$

or

$$D(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}) - P(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\})$$

In the first case, $S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}$ is non non-overlapping. This implies $D(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}) = 0$ (because two of the rows are idential) and $P(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}) = 0$ (because a triangle cannot be in two different lozenges at the same time). Thus, D and P agree. A similar argument holds in the second case. Thus, D and P agree whenever a lozenge of type II is added to S.

An identical holds for type III lozenges, except that we use (6) and (8) instead of (5) and (7). $\square$

We have been describing a lozenge as a pair (x, n, x', n'). It can also be described as (x', n', λ), where (x', n') is the location of the white triangle and $\lambda \in \{I, II, III\}$ is the type of the loznege. Thus the proposition can be restated as the following statement.

Corollary 2.7. *For any non-overlappign* $(x'_1, n'_1, \lambda_1), \ldots, (x'_k, n'_k, \lambda_k)$*,*

$$\begin{aligned}\mathbb{P}(&\textit{There is a lozenge } (x'_j, n'_j, \lambda_j) \textit{ at time } t \textit{ for each } j = 1, \ldots, k) \\ &= \det[K_\lambda(x'_i, n'_i, \lambda_i, x'_j, n'_j, t)]_{1\le i,j\le k},\end{aligned}$$

where

$$K_\lambda(x, n, \lambda, x', n', t) \;=\; \begin{cases} \tilde{K}(x, n, x', n', t), & \textit{when } \lambda = I \\ \tilde{K}(x - \delta_{n-1}, n-1, x', n', t), & \textit{when } \lambda = II \\ \tilde{K}(x - \delta_{n-1} - 1, n-1, x', n', t), & \textit{when } \lambda = III \end{cases}$$

Proof. This is a result of the correspondences

$$\begin{aligned} (x', n', I) &\text{ iff } (x', n', x', n'), \\ (x', n', II) &\text{ iff } (x' - \delta_{n'-1}, n'-1, x', n'), \\ (x', n', III) &\text{ iff } (x' - \delta_{n'-1} - 1, n'-1, x', n'). \end{aligned}$$

□

There are two different formulas for the height function. One formula is

$$h(x, n) = \sum_{s>x} \mathbf{1}(\text{lozenge of type I at } (s, n)). \tag{17}$$

It is possible to only use (17) to complete the proof. However, when there are multiple points on one level, i.e. not all $\eta_1, \ldots, \eta_k$ are distinct, the computation becomes much more complicated. This is because lozenges of type I will appear in multiple sums of the form (17). We can avoid this difficulty by introducing another formula for the height function:

$$h(x, n) = h(x + \delta_n + \delta_{n+1} + \ldots + \delta_{n'-1}, n') + H_{n,n'}(x), \tag{18}$$

where, for $n < n'$,

$$H_{n,n'}(x) = -\sum_{p=n+1}^{n'} \mathbf{1}(\text{lozenge of type II at } (x + \delta_n + \delta_{n+1} \ldots + \delta_{p-1}, p)). \tag{19}$$

Therefore, the expression

$$\mathbb{E}\left(\prod_{j=1}^{k}[h(x_j, n_j) - \mathbb{E}(h(x_j, n_j))]\right) \tag{20}$$

can be expressed as a sum of terms of the form

$$\mathbb{E}\left(\prod_{j=1}^{k'}[h(x_j,n_j)-\mathbb{E}(h(x_j,n_j))]\prod_{l=k'+1}^{k}[H_{n_l,n_l'}(x_l)-\mathbb{E}(H_{n_l,n_l'}(x_l))]\right). \tag{21}$$

Lemma 2.8. *Assume that the following sets are disjoint:*

$$\{(s,n_j):s>x_j\},\ 1\le j\le k'$$
$$\{(x_l+\delta_{n_l}+\delta_{n_l+1}\ldots+\delta_{p-1},p):n_l+1\le p\le n_l'\},\ k'+1\le l\le k.$$

Then

$$(21)=\sum_{s_1>x_1}\cdots\sum_{s_{k'}>x_{k'}}\sum_{p_{k'+1}=n_{k'+1}+1}^{n'_{k'+1}}\cdots\sum_{p_k=n_k+1}^{n'_k}\det\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \tag{22}$$

where the matrix blocks are:

$$\begin{aligned}
A_{11}&=[(1-\delta_{ij})\tilde{K}(s_i,n_i,s_j,n_j,t)]_{1\le i,j\le k'}\\
A_{12}&=[\tilde{K}(s_i,n_i,x_j,p_j,t)]_{1\le i\le k',\ k'+1\le j\le k}\\
A_{21}&=[-\tilde{K}(x_i-\delta_{p_i-1},p_i-1,s_j,n_j,t)]_{k'+1\le i\le k,\ 1\le j\le k'}\\
A_{22}&=[-(1-\delta_{ij})\tilde{K}(x_i-\delta_{p_i-1},p_i-1,x_j,p_j,t)]_{k'+1\le i,j\le k}
\end{aligned}$$

Proof. By applying Corollary 2.7 to (17) and (19), we see that

$$\mathbb{E}\left(\prod_{j=1}^{k'}h(x_j,n_j)\prod_{l=k'+1}^{k}H_{n_l,n_l'}(x_l)\right)$$

equals the right hand side of (22) with the $(1-\delta_{ij})$ terms removed. It is well-known that subtracting the expectation corresponds to putting zeroes on the diagonal. For example, this is noticed in the proof of Theorem 7.2 of [7]. $\square$

Write the determinant in (22) as a sum over permutations σ in S_k. If the cycle decomposition of σ contains the cycle $(c_1\ c_2\ \ldots\ c_r)$ of length r and M denotes the matrix in the right hand side of (22), then the contribution from σ is

$$\sum_{s_1}\cdots\sum_{s_{k'}}\sum_{p_{k'+1}}\cdots\sum_{p_k}\mathrm{sgn}(\sigma)M_{c_1c_2}M_{c_2c_3}\ldots M_{c_rc_1}(\cdots)(\cdots),$$

where $(\cdots)(\cdots)$ correspond to other cycles of σ. Let ψ_{c_ι} denote s_{c_ι} if $1\le c_\iota\le k'$, and p_{c_ι} if $k'<c_\iota\le k$. Since the sum over ψ_{c_ι} only affects the matrix terms $M_{c_{\iota-1}c_\iota}$ and $M_{c_\iota c_{\iota+1}}$, the contribution from σ is

$$\left((-1)^{r-1}\sum_{\psi_{c_1}}\cdots\sum_{\psi_{c_r}}M_{c_1c_2}M_{c_2c_3}\ldots M_{c_rc_1}\right)(\ldots), \tag{23}$$

where $(\ldots)$ denote other cycles. In other words, the contribution from σ can be expressed as a product over the cycles in the cycle decomposition of σ.

Note that if σ fixes any points, then the correponding contribution is zero because all the diagonal entries are zero.

2.3 Analysis steps in proof of Theorem 2.3

In (22), set $x_j = [N\nu_j], n_l = [N\eta_l]$, and $t = N\tau$. Our goal is to find the limit of (22) as $N \to \infty$. Expanding the determinant into a sum over $\sigma \in S_k$, we just saw that the contribution from a fixed σ is of the form (23). First note that if any of the ψ_{c_i} denotes p_{c_i}, then

$$\sum_{\psi_{c_1}} \cdots \sum_{\psi_{c_r}} M_{c_1c_2}M_{c_2c_3}\ldots M_{c_rc_1} \to 0.$$

This is because each $M_{c_jc_{j+1}}$ is proportional to $1/N$ (by Proposition 4.5, so $M_{c_1c_2}M_{c_2c_3}\ldots M_{c_rc_1}$ is proportional to N^{-r}, but the sum is only taken over $\mathcal{O}(N^{r-1})$ terms. Therefore, (20) can be expressed as a single term of the form in (21), and in this term $k' = k$.

Now we will prove (stated as Theorem 2.10 below) that

$$\sum_{s_{c_1}} \cdots \sum_{s_{c_r}} M_{c_1c_2}M_{c_2c_3}\ldots M_{c_rc_1} \to \left(\frac{1}{2\pi}\right)^r \int_{\bar{\Omega}_1}^{\Omega_1} dz_1 \cdots \int_{\bar{\Omega_r}}^{\bar{\Omega_r}} dz_r \frac{f(z_1,z_2)}{G'_\nu(z_1)} \cdots \frac{f(z_r,z_r)}{G'_\nu(z_r)}$$

Once this is proven, (11) implies that the total contribution from $S_k - \mathcal{F}_k$ equals zero. When $l = 2$, then the right hand side is just $\mathcal{G}(\Omega_1, \Omega_2)$, completing the proof of Theorem 2.3.

Recall the definitions of G and Ω from section 2.1. Set $\theta : \mathcal{D} \to [0, \pi)$ to be

$$\theta(\nu,\eta,\tau) = \frac{1}{2}\arg G''(\nu,\eta,\tau,\Omega(\nu,\eta,\tau)).$$

Proposition 2.9. *For $i = 1, 2, 3$, let $(\nu_i, \eta_i, \tau) \in \mathcal{D}$, $x_i = [N\nu_i]$, $n_i = [N\eta_i]$ and $t = N\tau$. For $i = 1, 3$, let $G_i(z)$ denote $G(\nu_i, \eta_i, \tau, z)$, let θ_i denote $\theta(\nu_i, \eta_i, \tau)$ and let Ω_i denote $\Omega(\nu_i, \eta_i, \tau)$. Let $\Gamma_+ := \{\Omega(\nu, \eta_2, \tau) : \nu_2 \le \nu < q_2(\eta_2, \tau)\}$ and $\Gamma_- = \bar{\Gamma}_+$. Let $G'_\nu(z) = (\partial^2/\partial z\partial v)G(\nu_2, \tau_2, \tau, z)$. Then*

$$\begin{aligned}
&\sum_{y>[N\nu_2]} K(x_1,n_1,y,n_2,t)K(y,n_2,x_3,n_3,t) \\
&\qquad = o\left(\frac{1}{N}\right) + \frac{e^{N\Re((G_1(\Omega_1)-G_3(\Omega_3)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_3''(\Omega_3)|}} \int_{\Gamma_+\cup\Gamma_-} \frac{dz}{2\pi G'_{\nu_1}(z)} \\
&\quad \times \Big[f(\Omega_1,z)f(z,\Omega_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}} + f(\bar{\Omega}_1,z)f(z,\Omega_3)\frac{e^{-iN\Im(G_1(\Omega_1))+i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}} \\
&+ f(\Omega_1,z)f(z,\bar{\Omega}_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{-iN\Im(G_3(\Omega_3))-i\theta_3}} + f(\bar{\Omega}_1,z)f(z,\bar{\Omega}_3)\frac{e^{-iN\Im(G_1(\Omega_1))+i\theta_1}}{e^{-iN\Im(G_3(\Omega_3))-i\theta_3}}\Big].
\end{aligned} \tag{24}$$

Proof. Let $G_2(z)$ denote $G([y/N], \eta_2, \tau, z)$, let θ_2 denote $\theta([y/N], \eta_2, \tau)$ and Ω_2 denote $\Omega(y/N, \eta_2, \tau)$. Fix some $\beta \in (-1/2, 0)$ and split up the sum into two parts: the first part is from $\lfloor N\nu_2 \rfloor$ to $\lfloor N(q_2 - N^\beta) \rfloor$, while the second sum is from $\lfloor N(q_2 - N^\beta) \rfloor$ to $\lfloor Nq_2 \rfloor$. Since there are no particles to the right of Nq_2 in the limit $N \to \infty$, the sum from Nq_2 to ∞ can be ignored. It is common to refer to the first sum as the bulk and the second sum as the edge. First examine the bulk. By Proposition 4.5,

$$
\begin{aligned}
&K(x_1, n_1, y, n_2, t)K(y, n_2, x_3, n_3, t) \\
&\quad = \frac{e^{N\Re((G_1(\Omega_1)-G_2(\Omega_2)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}} \frac{e^{N\Re((G_2(\Omega_2)-G_3(\Omega_3)))}}{2\pi N\sqrt{|G_2''(\Omega_2)|}\sqrt{|G_3''(\Omega_3)|}} \\
&\quad \times \left[f(\Omega_1, \Omega_2) f(\Omega_2, \Omega_3) \frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_2(\Omega_2))+i\theta_2}} \frac{e^{iN\Im(G_2(\Omega_2))-i\theta_2}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}} + \circlearrowleft \right] \\
&\qquad + \mathcal{O}(G_2''(\Omega_2)^{-4}N^{-3}) + \mathcal{O}(G_2''(\Omega_2)^{-7}N^{-4}), \quad (25)
\end{aligned}
$$

where $\circlearrowleft$ denotes the other fifteen terms that occur in the sum. First let us examine the error term in the bulk.

By (2) of Definition 2.1, each term in the error is bounded by $(N^{\beta/2})^{-4}N^{-3}$ and $(N^{\beta/2})^{-7}N^{-4}$, respectively. There are $\sim N$ terms, and since $\beta > -1/2$, we must have $-2\beta - 3 + 1 < -1$ and $-7\beta/2 - 4 + 1 < -1$. Therefore the sum is $o(1/N)$.

Now let us return to the main term in the bulk. For eight of the sixteen terms in $\circlearrowleft$, the expression $e^{iN\Im(G_2(\Omega_2))}$ cancels in the numerator and the denominator. By Propositions 4.2, these eight terms are $o(1/N)$. By 4.4, the other eight terms equal

$$
\begin{aligned}
&\frac{e^{N\Re((G_1(\Omega_1)-G_3(\Omega_3)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_3''(\Omega_3)|}} \int_{\nu_2}^{\infty} \frac{e^{-2i\theta_2}}{2\pi|G_2''(\Omega_2)|} \\
&\qquad \times \left[f(\Omega_1, \Omega_2) f(\Omega_2, \Omega_3) \frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}} + \dots \right] d\nu + o\left(\frac{1}{N}\right),
\end{aligned}
$$

where $\dots$ represent the other seven terms. Of the eight total terms, four have $f(\cdot, \Omega_2)f(\Omega_2, \cdot)$ and four have $f(\cdot, \bar{\Omega}_2)f(\bar{\Omega}_2, \cdot)$. For the four terms with the expression Ω_2, make the substitution $z = \Omega(\nu, \eta_2, \tau)$. The new integration path is Γ_+. By taking the partial of (9) with respect to ν and using the chain rule,

$$
\frac{\partial \Omega}{\partial \nu} = -\frac{G_\nu'(\Omega)}{G''(\Omega)},
$$

which implies

$$
\frac{e^{-2i\theta_2}}{2\pi|G_2''(\Omega_2)|} d\nu = \frac{d\nu}{2\pi G_2''(\Omega_2)} = -\frac{dz}{2\pi G_\nu'(z)}.
$$

For the four terms with $\bar{\Omega}_2$, make the substitution $z = \bar{\Omega}(\nu, \eta_2, \tau)$. The path of

integration is Γ_-. Finally, the integral becomes

$$o\left(\frac{1}{N}\right)+\frac{e^{N\Re((G_1(\Omega_1)-G_3(\Omega_3)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_3''(\Omega_3)|}}\int_{\Gamma_+\cup\Gamma_-}\frac{dz}{2\pi G_{\nu_1}'(z)}$$
$$\times\Big[f(\Omega_1,z)f(z,\Omega_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}}+f(\bar{\Omega}_1,z)f(z,\Omega_3)\frac{e^{-iN\Im(G_1(\Omega_1))+i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}}$$
$$+f(\Omega_1,z)f(z,\bar{\Omega}_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{-iN\Im(G_3(\Omega_3))-i\theta_3}}+f(\bar{\Omega}_1,z)f(z,\bar{\Omega}_3)\frac{e^{-iN\Im(G_1(\Omega_1))+i\theta_1}}{e^{-iN\Im(G_3(\Omega_3))-i\theta_3}}\Big].$$

Now we sum over the edge. By Proposition 4.11 and (2) of Definition 2.1, the sum is bounded above by

$$\sum_{y=(q_2-N^\beta)N}^{q_2N}|G_2(\Omega_2)^{-1}|N^{-2}\leq\sum_{y=0}^{N^{\beta+1}}\left(\frac{y}{N}\right)^{1/2}N^{-2}=\mathcal{O}(N^{3\beta/2-1}).$$

As long as $\beta<0$, the sum over the edge is also $o(1/N)$. □

Theorem 2.10. *For $i=1,\ldots,l$, let $(\nu_i,\eta_i,\tau)\in\mathcal{D}$ and set $x_i=[N\nu_i],n_i=N\eta_i$. For $i=1,\ldots,l$, let $G_i(z)$ denote $G(\nu_i,\eta_i,z)$, let θ_i denote $\theta(\nu_i,\eta_i,\tau)$ and let Ω_i denote $\Omega(\nu_i,\eta_i,\tau)$. Let $\Gamma_i^+:=\{\Omega(\nu,\eta_i,\tau):\nu_1\leq\nu<q_2(\eta_i,\tau)\}$ and $\Gamma_i^-=\bar{\Gamma}_i^+$. Then*

$$\sum_{y_1>[N\nu_1]}\cdots\sum_{y_l>[N\nu_l]}\prod_{i=1}^{l}K(y_i,x_i,y_{i+1},x_{i+1},t)$$
$$\to\left(\frac{1}{2\pi}\right)^l\int_{\Gamma_1^+\cup\Gamma_1^-}dz_1\cdots\int_{\Gamma_l^+\cup\Gamma_l^-}dz_l\frac{f(z_1,z_2)}{G_\nu'(z_1)}\cdots\frac{f(z_l,z_1)}{G_\nu'(z_l)}.$$

The indices are taken cyclically.

Proof. By Proposition 4.5, the product has 4^l terms. Each application of Proposition 2.9 decreases the number of terms by a factor of 4, so repeated applications of Proposition 2.9 yields the result. □

3 Specific Results

3.1 Particle system with a wall

We now return to the particle system with a reflecting wall described in the Introduction. For notational reasons, it is more convenient to use different coordinates. Instead of labeling the levels as $1,2,3,\ldots$, it is more convenient to label them as $(1,-1/2),(1,1/2),(2,-1/2),(2,1/2),\ldots$. If the (n_1,a_1) is at least as high as the (n_2,a_2) level, then this will be denoted as $(n_1,a_1)\trianglerighteq(n_2,a_2)$. This happens if and only if $2n_1+a_1\geq 2n_2+a_2$. Using the notation of Section

2.2, $m_{(n,a)} = n$ and $\delta_{(n,a)} = a + 1/2$. Along the horizontal direction, we will use a square lattice, so that the particles live on $\mathbb{N}$ instead of $2\mathbb{N}$ or $2\mathbb{N}+1$.

Let $m_{a_1}(dz)$ be defined by

$$m_{a_1}(dz) \begin{cases} \dfrac{dz}{2iz}, & a_1 = -1/2, \\ \dfrac{-(z^{1/2} - z^{-1/2})^2 dz}{4iz}, & a_1 = 1/2. \end{cases}$$

Let $\mathsf{J}_s^{(\pm 1/2, -1/2)}$ denote the (normalized) Jacobi polynomial with parameters $(\pm 1/2, -1/2)$. The normalization is set so that for any nonzero complex number z, $\mathsf{J}_s^{(\pm 1/2, -1/2)}$ satisfies

$$\mathsf{J}_s^{(-1/2,-1/2)} \left(\frac{z + z^{-1}}{2} \right) = \frac{z^s + z^{-s}}{2}, \tag{26}$$

$$\mathsf{J}_s^{(1/2,-1/2)} \left(\frac{z + z^{-1}}{2} \right) = \frac{z^{s+1/2} - z^{-s-1/2}}{z^{1/2} - z^{-1/2}}. \tag{27}$$

Let $W^{(a,-1/2)}(s)$ be defined for nonnegative integers s by

$$W^{(a,-1/2)}(s) = \begin{cases} 2, & \text{if} \quad s > 0, a = -\frac{1}{2}, \\ 1, & \text{if} \quad s = 0, a = -\frac{1}{2}, \\ 1, & \text{if} \quad s \geq 0, a = \frac{1}{2}. \end{cases}$$

Note that for $a = \pm 1/2$,

$$\frac{W^{(a,-1/2)}(s_1)}{\pi} \oint_{|z|=1} \mathsf{J}_{s_1}^{(a,-1/2)} \left(\frac{z + z^{-1}}{2} \right) \mathsf{J}_{s_2}^{(a,-1/2)} \left(\frac{z + z^{-1}}{2} \right) m_a(dz) = \delta_{s_1 s_2} \tag{28}$$

By Theorem 4.1 of [3], the correlation functions are determinantal with kernel

$$\begin{aligned} &K(n_1, a_1, s_1, n_2, a_2, s_2, t) \\ &= \frac{W^{(a_1,-1/2)}(s_1)}{2\pi^2 i} \oint \oint \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} \mathsf{J}_{s_1}^{(a_1,-1/2)} \left(\frac{z + z^{-1}}{2} \right) \mathsf{J}_{s_2}^{(a_2,-1/2)} \left(\frac{v + v^{-1}}{2} \right) \\ &\qquad \times \frac{(\frac{z+z^{-1}}{2} - 1)^{n_1}}{(\frac{v+v^{-1}}{2} - 1)^{n_2}} \frac{1 - v^{-2}}{z + z^{-1} - v - v^{-1}} m_{a_1}(dz) dv \end{aligned} \tag{29}$$

$$\begin{aligned} &+ \mathbf{1}_{(n_1,a_1) \trianglerighteq (n_2,a_2)} \left(\frac{W^{(a_1,-1/2)}(s_1)}{\pi} \oint \mathsf{J}_{s_1}^{(a_1,-1/2)} \left(\frac{z + z^{-1}}{2} \right) \mathsf{J}_{s_2}^{(a_2,-1/2)} \left(\frac{z + z^{-1}}{2} \right) \right. \\ &\qquad \left. \times \left(\frac{z + z^{-1}}{2} - 1 \right)^{n_1 - n_2} m_{a_1}(dz) \right), \end{aligned} \tag{30}$$

where the z-contour is the unit circle and the v-contour is a circle centered at the origin with radius bigger than 1.

Theorem 3.1. *The determinantal point process is normal. The Green's function is given by*

$$\mathcal{G}(z,w)=\frac{1}{2\pi}\log\left(\frac{z+z^{-1}-\bar{w}-\bar{w}^{-1}}{z+z^{-1}-w-w^{-1}}\right).$$

Once we prove the point process is normal, the expression for the Green's function follows from Theorem 2.3 with

$$G(\nu,\eta,\tau;u)=\tau\frac{u+u^{-1}}{2}+\eta\log\left(\frac{u+u^{-1}}{2}-1\right)-\nu\log u,$$

$$f(u,v)=\frac{1}{v}\frac{1-u^{-2}}{v+v^{-1}-u-u^{-1}}.$$

In section 3.2, we show that the third condition in Definition 2.1 is satisfied. In section 3.3, we show that the fourth and second conditions are satisfied. Since these are conditions are the hardest to prove, we will focus mainly on their proofs. The fifth conditions follows from the substitution $u_j=z_j+z_j^{-1}$ and (5) of Remark 2.2.

3.2 Algebraic steps in proof of theorem 3.1

Proposition 3.2. *Let $\mathcal{C}_0(n,a,s)$ equal*

$$\mathcal{C}_0(n,a,s)=\begin{cases}(-1)^s(-2)^{n-1}, & a=-1/2\\ (-1)^s(-2)^n, & a=1/2\end{cases}$$

and $c_0(n_1,a_1,s_1,n_2,a_2,s_2)=\mathcal{C}_0(n_1,a_1,s_1)/\mathcal{C}_0(n_2,a_2,s_2)$. Then $\tilde{K}=c_0K$ satisfies (3)–(8) *for $L=1$.*

Proof. Using (26)–(27) and the orthogonality relation (28), it is straightforward to check that (3) and (4) hold. What happens is that in the left hand side of (3) or (4), one obtains six terms, three of which come from (29) and three of which coe from (30). The three terms from (29) always sum to 0, while the three terms from (30) sum to 0 or 1.

Now we will prove (7)-(8) when $a_1=-1/2$. The term (30) equals zero, so we only need to look at (29). Explicitly, the expression is

$$K(n,-1/2,s,n,1/2,s',t)=\frac{2}{2\pi^2 i}\oint\oint_{|z|=1}\frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}}\left(\frac{z^s+z^{-s}}{2}\right)$$
$$\times\left(\frac{v^{s'+1/2}-v^{-s'-1/2}}{v^{1/2}-v^{-1/2}}\right)\frac{(\frac{z+z^{-1}}{2}-1)^n}{(\frac{v+v^{-1}}{2}-1)^n}\frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}}\frac{dzdv}{2iz},$$

and we want the asymptotic result when $s, s' \to \infty$ in such a way that $s - s'$ is 0 or 1. Expand the paranthetical expression $v^{s'+1/2} - v^{-s'-1/2}$ to get two terms, each of which is a double integral. Since $1 = |z| < |v|$, the term with $v^{-s'-1/2}$ goes to zero. For the remaining term, expand $z^s + z^{-s}$ to get two terms. For the term with z^s, make the substitution $z \mapsto z^{-1}$. What remains is

$$\frac{2}{2\pi^2 i} \oint \oint_{|z|=1} \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} \frac{v^{s'}}{z^s} \frac{v}{v-1} \frac{(\frac{z+z^{-1}}{2}-1)^n}{(\frac{v+v^{-1}}{2}-1)^n} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dzdv}{2iz}.$$

Now deform the z-contour to the circle $|z| = 1 + 2\epsilon$ and the v-contour to the circle $|v| = 1+\epsilon$, where $\epsilon > 0$. With these deformations, $|v| < |z|$, so the double integral goes to zero. However, residues are picked up when the contours pass through each other. These residues equal

$$-\frac{2}{\pi} \oint_{|z|=1+2\epsilon} z^{s'-s} \frac{z}{z-1} \frac{dz}{2iz}.$$

There is a residue at $z = 1$ which equals -2, and a residue at $z = 0$ which equals 0 for $s' \geq s$ and 2 for $s > s'$. Since $c_0(n, -1/2, s, n, 1/2, s) = -1/2$, this proves (7) and (8) when $a_1 = -1/2$. The case when $a_1 = 1/2$ is similar.

It remains to show (5) and (6). When considering the product of two kernels, we obtain a quadruple integral. After the substitutions $z_1 \mapsto z_1^{-1}$ and $v_2 \mapsto v_2^{-1}$, the part of the integrand that depends on s is just $(z_1/v_2)^s$. Therefore, deforming contours so that $|v_2| > |z_1|$ gives (5) and (6). $\square$

3.3 Analysis steps in proof of theorem 3.1

For this section, we need a slightly different expression for the kernel. By (40)–(42) of [3], the kernel equals

$$\begin{aligned} &K(n_1, a_1, s_1; n_2, a_2, s_2, t) \\ &= \frac{W^{(a_1,-1/2)}(s_1)}{2\pi^2 i} \int_{e^{-i\theta}}^{e^{i\theta}} \oint_{|z|=1} \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} \mathsf{J}_{s_1}^{(a_1,-1/2)} \left(\frac{z+z^{-1}}{2}\right) \mathsf{J}_{s_2}^{(a_2,-1/2)} \left(\frac{v+v^{-1}}{2}\right) \\ &\qquad \times \frac{(\frac{z+z^{-1}}{2}-1)^{n_1}}{(\frac{v+v^{-1}}{2}-1)^{n_2}} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} m_{a_1}(dz)dv \end{aligned} \tag{31}$$

$$+\mathbf{1}_{(n_1,a_1)\trianglerighteq(n_2,a_2)} \left(\frac{W^{(a_1,-1/2)}(s_1)}{\pi} \oint_{|z|=1} \mathsf{J}_{s_1}^{(a_1,-1/2)} \left(\frac{z+z^{-1}}{2}\right) \mathsf{J}_{s_2}^{(a_2,-1/2)} \left(\frac{z+z^{-1}}{2}\right) \times \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} m_{a_1}(dz) \right) \tag{32}$$

$$
+\left(\frac{W^{(a_1,-1/2)}(s_1)}{\pi}\int_{e^{-i\theta}}^{e^{i\theta}}\mathsf{J}_{s_1}^{(a_1,-1/2)}\left(\frac{z+z^{-1}}{2}\right)\mathsf{J}_{s_2}^{(a_2,-1/2)}\left(\frac{z+z^{-1}}{2}\right)\right.
$$
$$
\left.\times\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}m_{a_1}(dz)\right),\quad (33)
$$

where θ is any real number, and the arc from $e^{-i\theta}$ to $e^{i\theta}$ is outside the unit circle and does not cross $(-\infty,0]$.

Set

$$
G(\nu,\eta,\tau,z)=\tau\frac{z+z^{-1}}{2}+\eta\log\left(\frac{z+z^{-1}}{2}-1\right)-\nu\log z
$$

By Proposition 5.1.1 of [3], we can take $\mathcal{D}$ to be

$$
\mathcal{D}=\{(\nu,\eta,\tau):\eta,\tau>0,q_1(\eta,\tau)<\nu<q_2(\eta,\tau)\},
$$

where

$$
\begin{aligned}
q_1(\eta,\tau) &= \begin{cases}\eta\sqrt{-\dfrac{\tau^2}{2\eta^2}+\dfrac{5\tau}{\eta}+1-\dfrac{1}{2}\dfrac{\tau^2}{\eta^2}\left(1+\dfrac{4\eta}{\tau}\right)^{3/2}}, & 0\le\dfrac{\tau}{\eta}<\dfrac{1}{2},\\ 0, & \dfrac{1}{2}\le\dfrac{\tau}{\eta},\end{cases}\\
q_2(\eta,\tau) &= \eta\sqrt{-\frac{\tau^2}{2\eta^2}+\frac{5\tau}{\eta}+1+\frac{1}{2}\frac{\tau^2}{\eta^2}\left(1+\frac{4\eta}{\tau}\right)^{3/2}}.
\end{aligned}
$$

Lemma 3.3. *Let $\Omega_\pm$ denote $\Omega(\pm\nu,\eta,\tau)$. Then $\bar{\Omega}_+\Omega_-\equiv 1$.*

Proof. In general,

$$
G'(z)=\frac{p(z)}{r(z)},
$$

where p and r are

$$
\begin{aligned}
p(z) &= \tau+(2\eta+2\nu-\tau)z+(2\eta-2\nu-\tau)z^2+\tau z^3,\\
r(z) &= 2z^2(z-1).
\end{aligned}
$$

Let $p_\pm(z)$ denote the polynomial $p(z)$ corresponding to $(\pm\nu,\eta,\tau)$. Note that $z^3p_+(z^{-1})=p_-(z)$. By definition, $\Omega_\pm$ is the zero of $p_\pm$ that is in the upper half-plane. Therefore, $\Omega_-^{-1}=\bar{\Omega}_+$. □

Now let us return to the proof of the fourth condition in Definition 2.1. Start by examining (31). Expanding the parantheses, we obtain four terms corresponding to $z^{s_1}v^{s_2}, z^{s_1}v^{-s_2}, z^{-s_1}v^{-s_2}$, and $z^{-s_1}v^{s_2}$. For the two terms with z^{s_1}, make the substitution $z\to z^{-1}$. What remains are two terms, corresponding

to $z^{-s_1}v^{s_2}$ and $z^{-s_1}v^{-s_2}$. Therefore, (31) equals

$$\frac{W^{(-1/2,-1/2)}(s_1)}{4\pi^2 i}\int_{e^{-i\theta}}^{e^{i\theta}}\oint \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}}z^{-s_1}(v^{s_2}+v^{-s_2})$$
$$\times\frac{(\frac{z+z^{-1}}{2}-1)^{n_1}}{(\frac{v+v^{-1}}{2}-1)^{n_2}}\frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}}\frac{dzdv}{2iz}, \quad (34)$$

We now need to deform the contours in (34) to steepest descent paths. In other words, we need

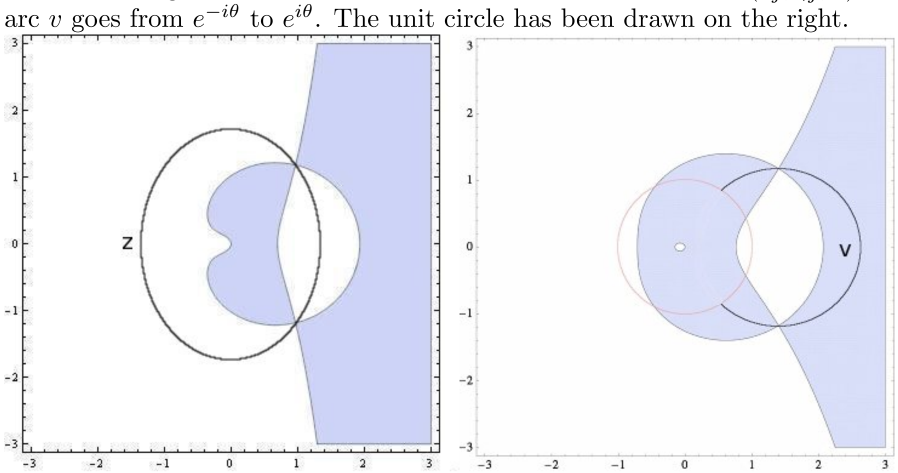
$$\Re(G(\nu_1,\eta_1,\tau,z)) < \Re(G(\nu_1,\eta_1,\tau,\Omega(\nu_1,\eta_1,\tau))) \quad (35)$$

for all z on the z-contour and

$$\Re(G(\nu_2,\eta_2,\tau,v)) > \Re(G(\nu_2,\eta_2,\tau,\Omega(\nu_2,\eta_2,\tau))), \quad (36)$$
$$\Re(G(-\nu_2,\eta_2,\tau,v)) > \Re(G(-\nu_2,\eta_2,\tau,\Omega(-\nu_2,\eta_2,\tau))) \quad (37)$$

for all v on the v-contour. By Lemma 3.3 and the definition of G, we see that $\Re(G(\nu_2,\eta_2,\tau,\Omega(\nu_2,\eta_2,\tau))) = \Re(G(-\nu_2,\eta_2,\tau,\Omega(-\nu_2,\eta_2,\tau)))$. If $|v|\geq 1$, then $\Re(G(-\nu_2,\eta_2,\tau,v)) \geq \Re(G(\nu_2,\eta_2,\tau,v))$. Since the steepest descent paths can go completely outside the unit circle (see Proposition 5.1.2 of [3]), (37) follows from (36).

Figure 5: On the left is $\Re(G(\nu_1,\eta_1,\tau,z)-G(\nu_1,\eta_1,\tau,\Omega(\nu_1,\eta_1,\tau)))$, and on the right is $\Re(G(\nu_2,\eta_2,\tau,v)-G(\nu_2,\eta_2,\tau,\Omega(\nu_2,\eta_2,\tau)))$. White regions indicate $\Re<0$ and shaded regions indicate $\Re>0$. The double zero occurs at $\Omega(\nu_j,\eta_j,\tau)$. The arc v goes from $e^{-i\theta}$ to $e^{i\theta}$. The unit circle has been drawn on the right.



If we deform the contours to the steepest descent paths Γ_1 and Γ_2 in Figure 5, we get that (31) asymptotically becomes

$$\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_1}\int_{\Gamma_2}\frac{\exp(NG(\nu_1,\eta_1,\tau,z))}{\exp(NG(\nu_2,\eta_2,\tau,v))}\frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}}\frac{dvdz}{z}$$

$$+\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{\exp(NG(\nu_1,\eta_1,\tau,z))}{\exp(NG(-\nu_2,\eta_2,\tau,v))} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dvdz}{z},$$

plus possibly the residues at $z = v$. Since Γ_2 goes outside the unit circle and the critical point of $G(-\nu_2, \eta_2, \tau, v)$ lies inside the unit circle, the second double integral is negligible.

Now we need to compute the possible residues at $z = v$. If the contours pass through each other, then the residues at $z = v$ equal

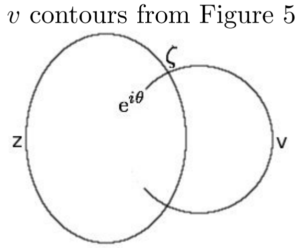
$$\frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i} \int_{e^{i\theta}}^{\zeta} z^{s_2-s_1} \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} \frac{dz}{z} + \frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i} \int_{\bar{\zeta}}^{e^{-i\theta}} z^{s_2-s_1} \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} \frac{dz}{z} \quad (38)$$

$$+\frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i} \int_{e^{i\theta}}^{\zeta} z^{-s_2-s_1} \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} \frac{dz}{z} + \frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i} \int_{\bar{\zeta}}^{e^{-i\theta}} z^{-s_2-s_1} \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} \frac{dz}{z}, \quad (39)$$

where ζ is any complex number satisfying (35) and (36). See Figure 6. If the contours do not pass through each other, then there is no contribution from the residues. For notational convenience, set

$$\xi = \begin{cases} \zeta, & \text{if } \zeta \text{ exists,} \\ e^{i\theta}, & \text{otherwise.} \end{cases}$$

Figure 6: The z and v contours from Figure 5. They intersect at ζ.



It is important to note that ξ is arbitrarily selected. The only requirement on ζ is that it satisfies the inequalities (35) and (36), and the only requirement on $e^{i\theta}$ is that $\Re(G_2(e^{i\theta})) > \Re(G_2(\Omega_2))$. So there exists $\epsilon > 0$ such that if $|\xi_1 - \xi| < \epsilon$, then ξ_1 also satisfies those inequalities.

Now we need to compute (32) and (33). Expanding the parantheses, we get four terms corresponding to $z^{s_1+s_2}, z^{s_1-s_2}, z^{s_2-s_1}, z^{-s_1-s_2}$. For the terms

corresponding $z^{-s_2-s_1}$ and $z^{s_1-s_2}$, make the substitution $z \to z^{-1}$. Therefore, the sum of (32),(33),(38),(39) equals

$$\frac{1}{4\pi i}\int_{\bar{\xi}}^{\xi} z^{s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z} \\ + \frac{1}{4\pi i}\int_{\bar{\xi}}^{\xi} z^{-s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z}, \quad (40)$$

where the contour crosses $(0,\infty)$ if $n_1 \geq n_2$, and it crosses $(-\infty,0)$ if $n_1 < n_2$. For each integral, deform the contour to a circular arc of constant radius. Then the absolute value of the integrand is maximized at the endpoints. To see this, note that $|z|$ is constant on the arc, so it suffices to analyze

$$R(x,y) := \left|\frac{(x+iy)+(x+iy)^{-1}}{2}-1\right|^2.$$

Now note that

$$\nabla R(x,y)\cdot(y,-x) = -\frac{y((x-1)^2+y^2)}{x^2+y^2}.$$

Therefore, R increases as we move counterclockwise along the arc in the upper half-plane, and R decreases as we move counterclockwise along the arc in the lower half-plane. If $n_1 \geq n_2$, then $R^{(n_1-n_2)/2}$ is maximized when R is maximized, which occurs at the endpoints since the contour crosses $(0,\infty)$. Likewise, if $n_1 < n_2$, then R is minimized at the endpoints. Thus, in both cases, $R^{(n_1-n_2)/2}$ is maximized at the endpoints.

Using a standard asymptotic analysis (see e.g. chapter 3 of [10]), we get that the asymptotic expansion of (40) is

$$\frac{c_1}{N}\xi^{N(\nu_2-\nu_1)}\left(\frac{\xi+\xi^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)} + \frac{\overline{c_1}}{N}\bar{\xi}^{N(\nu_2-\nu_1)}\left(\frac{\bar{\xi}+\bar{\xi}^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)} \\ + \frac{c_2}{N}\xi^{N(-\nu_2-\nu_1)}\left(\frac{\xi+\xi^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)} + \frac{\overline{c_2}}{N}\bar{\xi}^{N(-\nu_2-\nu_1)}\left(\frac{\bar{\xi}+\bar{\xi}^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)}$$

for some constants c_1, c_2. To complete the proof, notice that if

$$\left|\xi^{\pm\nu_2-\nu_1}\left(\frac{\xi+\xi^{-1}}{2}-1\right)^{\eta_1-\eta_2}\right| > e^{\Re(G_1(\Omega_1)-G_2(\Omega_2))}$$

for some selection of $\pm$, then the asymptotic expansion of the kernel would depend on ξ. But ξ was arbitrarily selected, so this is impossible.

Now that the fourth condition has been proved, it remains to show that the second condition in Definition 2.1 holds. Recall that $\Omega(\nu,\eta,\tau)$ is the root of $p(\nu,\eta,\tau,z)$ that lies in the upper half-plane, where p is the polynomial from Lemma 3.3. We thus need to solve

$$p(q_2(\eta,\tau)-\epsilon_1,\eta,\tau,\Omega(q_2(\eta,\tau),\eta,\tau)+\epsilon_2) = 0.$$

Since $\Omega(q_2(\eta,\tau),\eta,\tau)$ is a double zero of $p(q_2(\eta,\tau),\eta,\tau,z)$, we thus have to solve

$$\frac{1}{2}\epsilon_2^2 p''(\Omega) - 2\epsilon_1(\Omega + \epsilon_2 - \epsilon_2^2 - 2\epsilon_2\Omega - \Omega^2) + \mathcal{O}(\epsilon_2^3) = 0,$$

which implies that $\epsilon_2 = \mathcal{O}(\epsilon_1^{1/2})$. In other words, as ν approahces $q_2(\eta,\tau)$, $\Omega(\nu,\eta,\tau) - \Omega(q_2(\eta,\tau),\eta,\tau) = \mathcal{O}((q_2(\eta,\tau)-\nu)^{1/2})$. Plugging this into the expression for G'' gives the result.

4 Asymptotic Lemmas

4.1 Riemannian Approximations

Lemma 4.1. *Suppose that $g \in C^1[a,b]$ and $I \in C^2[a,b]$. Suppose that as $\delta \to 0$, the Lesbesgue measure of the set $\{x \in [a,b] : I'(x) \in 2\pi\mathbb{Z} + [-\delta,\delta]\}$ is $\mathcal{O}(\delta^a)$ for some positive a. Let $\epsilon_N \in [-1,1]$ depend on N. Then*

$$\lim_{N\to\infty} \sum_{k=1}^{\lfloor (b-a)N \rfloor} e^{iNI(a+(k+\epsilon_N)/N)} g\left(a + \frac{k+\epsilon_N}{N}\right)\frac{1}{N} = o(1).$$

Proof. Let t_k denote $a + (k+\epsilon_N)/N$. Note that $|t_k - t_s| = |k-s|/N$. Fix some $1 + N^{1/3} \le s \le \lfloor (b-a)N \rfloor - N^{1/3}$ and consider

$$\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g(t_k)\frac{1}{N}.$$

We bound this sum in two cases.

Case 1. Assume $I'(t_s) \notin 2\pi\mathbb{Z} + [-\delta,\delta]$. For $s - N^{1/3} \le k \le s + N^{1/3}$, Taylor's theorem says that

$$I(t_k) = [I(t_s) + I'(t_s)(t_k - t_s)] + [\frac{1}{2}I''(c_k)(t_k-t_s)^2] =: [I_1(t_k)] + [I_2(t_k)]$$

for some c_k between t_s and t_k. We will prove that

$$\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g(t_k)\frac{1}{N} \le \frac{99\delta^{-1}\|g\|_\infty}{N} + \frac{18\|g\|_\infty\|I''\|_\infty}{N} + \frac{3\|g'\|_\infty}{N^{4/3}}$$

Using the inequality

$$|g(t_k) - g(t_s)| \le \|g'\|_\infty \cdot |t_k - t_s| = \|g'\|_\infty \frac{|k-s|}{N},$$

we have that

$$\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)}(g(t_k) - g(t_s))\frac{1}{N}\right| \le \sum_{k=s-N^{1/3}}^{s+N^{1/3}} \|g'\|_\infty \frac{|k-s|}{N^2}$$
$$= 2\|g'\|_\infty \frac{N^{1/3}(N^{1/3}+1)}{N^2}. \quad (41)$$

Furthermore, for $|k-s| \le N^{1/3}$,

$$|1-e^{iNI_2(t_k)}| = |1-e^{iI''(c_k)(k-s)^2/(2N)}| \le |1-e^{i\|I''\|_\infty N^{-1/3}}| \le 9\|I''\|_\infty N^{-1/3}. \tag{42}$$

Also,

$$\begin{aligned}\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI_1(t_k)}\right| &= \left|e^{iNI(t_s)} \sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iI'(t_s)(k-s)}\right| \\ &\le \left|\frac{4}{e^{iI'(t_s)}-1}\right| \le 99\delta^{-1} \end{aligned} \tag{43}$$

Using (41), the definition of I_1 and I_2, (42) and (43) respectively,

$$\begin{aligned}\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g(t_k) \frac{1}{N}\right| &\le \left|g(t_s) \sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} \frac{1}{N}\right| + 3\|g'\|_\infty N^{-4/3} \\ &\le \|g\|_\infty \left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} \frac{e^{iNI_1(t_k)} + e^{iNI_1(t_k)}(e^{iNI_2(t_k)}-1)}{N}\right| + \frac{3\|g'\|_\infty}{N^{4/3}} \\ &\le \|g\|_\infty \left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} \frac{e^{iNI_1(t_k)}}{N}\right| + \frac{18\|g\|_\infty\|I''\|_\infty}{N} + \frac{3\|g'\|_\infty}{N^{4/3}} \\ &\le \frac{99\delta^{-1}\|g\|_\infty}{N} + \frac{18\|g\|_\infty\|I''\|_\infty}{N} + \frac{3\|g'\|_\infty}{N^{4/3}}\end{aligned}$$

Case 2. Assume that $I'(t_s) \in 2\pi\mathbb{Z} + (-\delta, \delta)$. In this case, only a simple estimate is needed:

$$\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g(t_k) \frac{1}{N}\right| \le \frac{2\|g\|_\infty}{N^{2/3}}.$$

Since the estimate in case 1 is much better than the estimate in case 2, we need an upper bound on how frequently case 2 can occur. In other words, we need an upper bound on the measure of the set $\{x \in [a,b] : I'(x) \in 2\pi\mathbb{Z}+(\delta,\delta)\}$. We assumed that

$$|\{x \in [a,b] : I'(x) \in 2\pi\mathbb{Z} + [\delta,\delta]\}| = \mathcal{O}(\delta^a).$$

Now we need to sum over all s in the set $\{N^{1/3}+1, 3N^{1/3}+1, 5N^{1/3}+1 \ldots, (b-a)N - N^{1/3}\}$. There are $\mathcal{O}(\delta^a N^{2/3})$ terms for which case 2 applies. Therefore,

$$\sum_{k=1}^{\lfloor (b-a)N \rfloor} e^{iNI(a+(k+\epsilon_N)/N)} g\left(a + \frac{k+\epsilon_N}{N}\right) \frac{1}{N} = \mathcal{O}\left(\delta^{-1} N^{-1/3}\right) + \mathcal{O}(\delta^a),$$

and setting $\delta = N^{-1/6}$ yields the result. □

Proposition 4.2. *Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ is a function such that for each $t > 0$,*

$$f(\lfloor tN \rfloor) = e^{iNI(t)} g(t) N^d + o(N^d) \text{ as } N \to \infty,$$

where g and I satisfy the same assumptions as in Lemma 4.1. Further suppose that the error term $o(N^d)$ is uniform, i.e.

$$\frac{f(\lfloor tN \rfloor) - e^{iNI(t)} g(t) N^d}{N^d} \to 0 \text{ uniformly on } [a, b].$$

Then as $N \to \infty$,

$$\sum_{x=\lfloor aN \rfloor + 1}^{\lfloor bN \rfloor} f(x) = o(N^{d+1}).$$

Proof. Let $\epsilon > 0$. By assumption, there exists N_0 such that for $N > N_0$ and $x \in [\lfloor aN \rfloor + 1, \lfloor bN \rfloor]$

$$\left| f(x) - e^{iNI(x/N)} g\left(\frac{x}{N}\right) N^d \right| < \epsilon N^d.$$

By summing over x, we see that

$$\left| \sum_{x=\lfloor aN \rfloor + 1}^{\lfloor bN \rfloor} f(x) - e^{iNI(x/N)} g\left(\frac{x}{N}\right) N^d \right| \leq (b-a)\epsilon N^{d+1}.$$

The Proposition now follows from Lemma 4.1. □

Lemma 4.3. *Suppose g is a function on $[a, b]$ with bounded variation. Then as $N \to \infty$,*

$$\sum_{x=\lfloor aN \rfloor + 1}^{\lfloor bN \rfloor} g\left(\frac{x}{N}\right) = N \int_a^b g(t)dt + o(N).$$

Proof. Let $\epsilon > 0$. By the definition of a Riemann sum, there exists M such that

$$\left| \int_a^b g(t)dt - \sum_{k=1}^{\lfloor (b-a)N \rfloor} g\left(a + \frac{k}{N}\right) \frac{1}{N} \right| < \epsilon/3 \text{ for all } N > M.$$

Furthermore, for $N > 3BV(g)/\epsilon$,

$$\left| \frac{1}{N} \sum_{k=1}^{\lfloor (b-a)N \rfloor} \left(g\left(a + \frac{k}{N}\right) - g\left(a + \frac{k}{N} - \frac{\{aN\}}{N}\right)\right) \right| \leq \frac{BV(g)}{N} < \epsilon/3.$$

Finally, for $N > 3\|g\|_\infty/\epsilon$,

$$\left| \frac{1}{N} \sum_{k=1}^{\lfloor (b-a)N \rfloor} g\left(a + \frac{k}{N} - \frac{\{aN\}}{N}\right) - \frac{1}{N} \sum_{\lfloor aN \rfloor + 1}^{\lfloor bN \rfloor} g\left(\frac{x}{N}\right) \right|$$
$$\leq g\left(\frac{\lfloor bN \rfloor}{N}\right) \frac{1}{N} \leq \frac{\|g\|_\infty}{N} < \epsilon/3.$$

□

Proposition 4.4. *Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is a function such that for each $t > 0$,*

$$f(\lfloor tN \rfloor) = g(t)N^d + o(N^d) \text{ as } N \to \infty,$$

where g is a function on $[a, b]$ of bounded variation. Further suppose that the error term $o(N^d)$ is uniform, i.e.

$$\frac{f(\lfloor tN \rfloor) - g(t)N^d}{N^d} \to 0 \text{ uniformly on } [a, b].$$

Then

$$\sum_{x=\lfloor aN \rfloor+1}^{\lfloor bN \rfloor} f(x) = N^{d+1} \int_a^b g(t)dt + o(N^{d+1}).$$

Proof. Let $\epsilon > 0$. By assumption, there exists N_0 such that for $N > N_0$ and $x \in [\lfloor aN \rfloor + 1, \lfloor bN \rfloor]$

$$g\left(\frac{x}{N}\right) N^d - \epsilon N^d \leq f(x) \leq g\left(\frac{x}{N}\right) N^d + \epsilon N^d.$$

We can see this by noting that when $t = (\lfloor aN \rfloor + 1)/N$, $\lfloor tN \rfloor = \lfloor aN \rfloor + 1$. Furthermore, increasing x by 1 causes t to increase by $1/N$, which causes $\lfloor tN \rfloor$ to increase by 1. By summing over x, we see that

$$N^d \sum_{x=\lfloor aN \rfloor+1}^{\lfloor bN \rfloor} (g\left(\frac{x}{N}\right) - \epsilon) \leq \sum_{x=\lfloor aN \rfloor+1}^{\lfloor bN \rfloor} f(x) \leq N^d \sum_{x=\lfloor aN \rfloor+1}^{\lfloor bN \rfloor} (g\left(\frac{x}{N}\right) + \epsilon)$$

The Proposition now follows from Lemma 4.3. □

4.2 Asymptotics

Proposition 4.5. *For $j = 1, 2$, let $(\nu_j, \eta_j, \tau) \in \mathcal{D}$, Ω_j denote $\Omega(\nu_j, \eta_j, \tau)$, $G_j(z)$ denote $G(\nu_j, \eta_j, \tau, z)$, and θ_j denote $\theta(\nu_j, \eta_j, \tau)$. With the assumptions from section 2.1,*

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{\exp(NG(\eta_1, \nu_1, \tau, u))}{\exp(NG(\eta_2, \nu_2, \tau, w))} f(u, w) dw du \\
&\quad = \mathcal{O}\left(\frac{G_1''(\Omega_1)^{-3} + G_2''(\Omega_2)^{-3}}{G_1''(\Omega_1)^{1/2} G_2''(\Omega_2)^{1/2}} N^{-2}\right) + \mathcal{O}(G_1''(\Omega_1)^{-7/2} G_2''(\Omega_2)^{-7/2} N^{-3}) \\
&+ \frac{e^{N\Re((G_1(\Omega_1) - G_2(\Omega_2)))}}{2\pi N \sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}} \times \Big[f(\Omega_1, \Omega_2) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{iN\Im(G_2(\Omega_2)) + i\theta_2}} + f(\Omega_1, \bar{\Omega}_2) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{-iN\Im(G_2(\Omega_2)) - i\theta_2}} \\
&\qquad + f(\bar{\Omega}_1, \Omega_2) \frac{e^{-iN\Im(G_1(\Omega_1)) + i\theta_1}}{e^{iN\Im(G_2(\Omega_2)) + i\theta_2}} + f(\bar{\Omega}_1, \bar{\Omega}_2) \frac{e^{-iN\Im(G_1(\Omega_1)) + i\theta_1}}{e^{-iN\Im(G_2(\Omega_2) - i\theta_2)}} \Big].
\end{aligned}$$

Proof. First, we show that the main term is correct.

By assumption, we can deform Γ_1 and Γ_2 so that Γ_j passes through $\Omega_j, \bar{\Omega}_j$ for $j = 1, 2$. The contributions to the integral away from $\Omega_j, \bar{\Omega}_j$ are exponentially

small, so we can replace Γ_j with $\gamma_j \cup \bar{\gamma}_j$, where γ_j and $\bar{\gamma}_j$ are steepest descent paths near Ω_j and $\bar{\Omega}_j$, respectively. The integration over $u \in \gamma_1 \cup \bar{\gamma}_1, w \in \gamma_2 \cup \bar{\gamma}_2$ expands into four integrations corresponding to $(u, w) \in \gamma_1 \times \gamma_2, \bar{\gamma}_1 \times \gamma_2, \gamma_1 \times \bar{\gamma}_2, \bar{\gamma}_1 \times \bar{\gamma}_2$. We explicitly do the calculation for $\gamma_1 \times \gamma_2$. The other three calculations are essentially identical.

Make the substitutions $s = G_1(\Omega_1) - G_1(u)$ and $t = G_2(\Omega_2) - G_2(w)$. In the neighborhood of $u = \Omega_1$ and $w = \Omega_2$, we have

$$f(u,w) \approx f(\Omega_1, \Omega_2), \quad s = -\frac{(u-\Omega_1)^2}{2} G_1''(\Omega_1), \quad t = -\frac{(w-\Omega_2)^2}{2} G_2''(\Omega_2),$$

which imply

$$\begin{aligned} G_1'(u) &= -\frac{ds}{du} = (u - \Omega_1) G_1''(\Omega_1) = \sqrt{-2s G_1''(\Omega_1)}, \\ G_2'(w) &= -\frac{dt}{dw} = (w - \Omega_2) G_2''(\Omega_2) = \sqrt{-2t G_2''(\Omega_2)}. \end{aligned}$$

Then we get

$$\begin{aligned} &\left(\frac{1}{2\pi i}\right)^2 e^{N(G_1(\Omega_1) - G_2(\Omega_2))} \int_0^\infty \int_0^\infty e^{-N(s+t)} \frac{f(u,w)}{G_1'(u) G_2'(w)} dt ds \\ &= 4 \cdot \frac{e^{N(G_1(\Omega_1) - G_2(\Omega_2))}}{8\pi^2 \sqrt{G_1''(\Omega_1)} \sqrt{G_2''(\Omega_2)}} f(\Omega_1, \Omega_2) \Big(\int_0^\infty s^{-1/2} e^{-Ns} ds \Big) \Big(\int_0^\infty t^{-1/2} e^{-Nt} dt \Big) \\ &= \frac{e^{N(G_1(\Omega_1) - G_2(\Omega_2))}}{2\pi N \sqrt{G_1''(\Omega_1)} \sqrt{G_2''(\Omega_2)}} f(\Omega_1, \Omega_2) \\ &= \frac{e^{N\Re((G_1(\Omega_1) - G_2(\Omega_2)))}}{2\pi N \sqrt{|G_1''(\Omega_1)|} \sqrt{|G_2''(\Omega_2)|}} \Big[f(\Omega_1, \Omega_2) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{iN\Im(G_2(\Omega_2)) + i\theta_2}} \Big], \end{aligned}$$

where the last equality follows from $G(\bar{z}) = \overline{G(z)}$. The 4 appears because the maps $u \mapsto s$ and $w \mapsto t$ are both two-to-one.

It still remains to show that the error term is correct. The remainder of this section is devoted to proving this. The idea is to reduce the double integral to progressively simpler forms. First, by a reparametrization, the integral over two arcs in $\mathbb{C}$ can be written as a integral in $\mathbb{R}^2$. Second, by using a Taylor approximation, the integral in $\mathbb{R}^2$ can be written as a product of two integrals in $\mathbb{R}$, each of which is of the form $\int e^{-NR(t)} \phi(t) dt$, where $R(t)$ has a maximum $t_{\max}$ in the interval of integration. Third, by using the implicit function theorem, this integral reduces to the form $\int e^{-Nt^2} g(t)$, where the interval of integration is a small neighbourhood $t_{\max}$. Fourth, this last integral is a slight generalization of $\int e^{-Nt} g(t) dt$, which is dealt with by the well-known Watson's lemma (Lemma 4.6 below). Since the first two steps have been done before (see Chapters 3 and 4 of [10]), we will focus mostly on the third and fourth steps.

Lemma 4.6. *Suppose $T > 0$ and $\phi(t)$ is a complex valued, absolutely integrable function on $[0, T]$:*

$$\int_0^T |\phi(t)| dt < \infty.$$

Also suppose that $\phi(t) = t^\sigma g(t)$, where $\sigma > -1$ and g is continuously differentiable in some neighbourhood of $t = 0$. Then for any $N > 1$ and $s \in [0, T]$,

$$\left|\int_0^T e^{-Nt}\phi(t)dt - \frac{g(0)\Gamma(\sigma+1)}{N^{\sigma+1}}\right| \leq \frac{\Gamma(\sigma+2)\sup\limits_{0\leq\tau\leq s}|g'(\tau)|}{N^{\sigma+2}} + e^{-Ns}\int_s^T |\phi(t)|dt$$
$$+ g(0)\int_s^\infty e^{-Nt}t^\sigma dt + e^{-Ns/2}\Gamma(3+2\sigma)\sup_{0\leq\tau\leq s}|g'(\tau)|$$

Proof. See Proposition 2.1 of [10]. □

Lemma 4.7. *Suppose $T > 0$ and $\phi(t)$ is a complex valued, absolutely integrable function on $[0, T]$:*

$$\int_0^T |\phi(t)|dt < \infty.$$

Also suppose that ϕ is twice continuously differentiable in some neighbourhood of $t = 0$. Then for any $N > 1$ and $s \in [0, m^2]$,

$$\left|\int_{-\alpha}^{\beta} e^{-Nt^2}\phi(t)dt - \frac{\phi(0)\sqrt{\pi}}{N^{1/2}}\right| \leq \frac{\sqrt{\pi}}{2}\frac{\sup\limits_{0\leq\tau\leq s}|g'(\tau)|}{N^{3/2}} + e^{-Ns}\int_s^{m^2}|\phi(t)|dt$$
$$+ e^{-Nm^2}\int_m^{\max(\alpha,\beta)}|\phi(t)|dt + \phi(0)\int_s^\infty e^{-Nt}t^{-1/2}dt + e^{-Ns/2}\sup_{0\leq\tau\leq s}|g'(\tau)|,$$

where $m = \min(\alpha, \beta)$ and

$$g(\tau) = \frac{1}{2}\left(\phi(\tau^{1/2}) + \phi(-\tau^{1/2})\right).$$

Proof. See pages 58–60 of [10]. □

Lemma 4.8. *Suppose that R and ϕ are infinitely continuously differentiable in some neighbourhood of t_{max}. Also suppose that t_{max} is a local maximum of R and $R''(t_{max}) < 0$. Then for any $N > 1$ and $s \in [0, m^2]$,*

$$\left|\int_{t_{max}-\delta_1}^{t_{max}+\delta_2} e^{NR(t)}\phi(t)dt - \phi(t_{max})e^{NR(t_{max})}\sqrt{\frac{-2\pi}{NR''(t_{max})}}\right| \leq \frac{\sqrt{\pi}}{2}\frac{\sup\limits_{0\leq\tau\leq s}|g'(\tau)|}{N^{3/2}}$$
$$+ e^{-Ns}\int_s^{m^2}|h(t)|dt + e^{-Nm^2}\int_m^{\max(\alpha,\beta)}|h(t)|dt + h(0)\int_s^\infty e^{-Nt}t^{-1/2}dt$$
$$+ e^{-Ns/2}\sup_{0\leq\tau\leq s}|g'(\tau)|,$$

where

$$\alpha = \sqrt{-R(t_{max}-\delta_1)},\ \beta = \sqrt{-R(t_{max}+\delta_2)},\ m = \min(\alpha, \beta),$$

$$h(s) = \phi(t_{max} + sv(s))(sv'(s) + v(s)),\ \ g(s) = \frac{1}{2}(h(s^{1/2}) + h(-s^{1/2})),$$

where $v(s)$ is an infinitely differentiable function solving

$$-R(t_{max}) + R(t_{max} + sv(s)) = -s^2. \tag{44}$$

Before continuing, a few estimates on $v(s)$ are needed.

Lemma 4.9. *Let $v(s)$ be as in* (44).

(a)

$$R''(t_{max}) = -2v(0)^{-2}. \tag{45}$$

$$v'(0) = \frac{R'''(t_{max})}{12} v(0)^4 \tag{46}$$

(b) Set $B = \sup |R^{(4)}/24|$. Then

$$|v(s) - v(0) - v'(0)s| < \left(\frac{5}{144} R'''(t_{max})^2 v(0)^7 + Bv(0)^5\right) s^2.$$

for

$$s < \min(|53R'''(t_{max})|/(625Bv(0)), (|R'''(t_{max})|v(0)^3)^{-1}, (50\sqrt{B}v(0)^2)^{-1}).$$

In particular, $|v(s) - v(0)| < |R'''(t_{max})|v(0)^4|s|/4$ and $|v(s) - v(0)| < v(0)/4$.

(c) Let

$$a_3 := \left(\frac{157}{16} Bv(0)^3 + \frac{101}{288} R'''(t_{max})^2 v(0)^5\right).$$

Then

$$|v(s) + sv'(s) - v(0) - 2v'(0)s| < \left(\frac{39}{32} R'''(t_{max})^2 v(0)^7 + \frac{471}{16} Bv(0)^5\right) s^2$$

$$\text{for}\ \ |s| < \min\left(\frac{|53R'''(t_{max})|}{625Bv(0)}, \frac{1}{50\sqrt{B}v(0)^2}, \frac{1}{6R'''(t_{max})v(0)^3}, \left|\sqrt{\frac{2v(0)^{-1}}{3a_3}}\right|\right). \tag{47}$$

(d) With the same bounds on $|s|$,

$$|2v'(s) + sv''(s) - 2v'(0)| < 450000(R'''(t_{max})^2 v(0)^7 + Bv(0)^5)s$$

Proof. (a) The proof comes from page 69 of [10]. It follows immediately from using implicit differentiation of (44) and setting $s = 0$.

(b) First notice that if $R_-(t) \leq R(t) \leq R_+(t)$ with $R_-(t_{\max}) = R(t_{\max}) = R_+(t_{\max})$ and $v_\pm$ are the solutions to $-R(t_{\max}) + R(t_{\max} \pm sv_\pm(s)) = -s^2$, then $v_- \leq v \leq v_+$. We will use

$$R_\pm(t) = R(t_{\max}) + \frac{1}{2}R''(t_{\max})(t - t_{\max})^2 + \frac{1}{6}R'''(t_{\max})(t - t_{\max})^3 \pm B(t - t_{\max})^4$$

Therefore, we obtain bounds on $v(s)$ by solving $-R(t_{\max}) + R(t_{\max} \pm sv_{\pm}(s)) = -s^2$, which is equivalent to solving

$$Q_{\pm,s}(y) := 1 - y_0^{-2}y^2 + Asy^3 \pm Bs^2y^4 = 0, \quad A = R'''(t_{\max})/6, \quad y_0 = v(0).$$

In other words $Q_{\pm,s}(v(s)) = 0$. We will use the intermediate value theorem to estimate roots of $Q_{\pm,s}$.

For $\epsilon = Ay_0^4 s/2 + (5A^2y_0^7/4 + By_0^5)s^2$ and $|s| = (HAy_0^3)^{-1}$ where H is any real number, we have

$$\begin{aligned} Q_{+,s}(y_0+\epsilon) \le \frac{1}{256A^{10}H^{10}y_0^{10}}(&256B^5 + 256A^2B^4p_2(H)y_0^2 + 32A^4B^3p_4(H)y_0^4 \\ &+ 16A^6B^2p_6(H)y_0^6 + A^8Bp_8(H)y_0^8 + 4A^{10}H^3p_5(H)y_0^{10}) \end{aligned}$$

where the p_i are polynomials which satisfy the following inequalities when $|H| > 6$

$$\begin{aligned} &p_2(H) = (5 + 2H + 4H^2) < 11H^2 \\ &p_4(H) = (75 + 60H + 132H^2 + 56H^3 + 48H^4) < 371H^4 \\ &p_6(H) = (125 + 150H + 360H^2 + 308H^3 + 312H^4 + 144H^5 + 48H^6) < 1447H^6 \\ &p_8(H) = (625 + 1000H + 2600H^2 + 3760H^3 + 4336H^4 + 4160H^5 + 1792H^6 \\ &\qquad + 1024H^7 - 256H^8) < -10H^8 \\ &H^3p_5(H) = H^3(125 + 150H + 360H^2 + 148H^3 + 208H^4 - 80H^5) < 0 \end{aligned}$$

Now setting $H := hy_0^{-1}A^{-1}\sqrt{B}$ where $h > 50$,

$$Q_{+,s}(y_0+\epsilon) < \frac{128 + 1408h^2 + 5936h^4 + 11576h^6 - 5h^8}{128h^{10}} < 0.$$

Since

$$Q_{+,s}(y_0 + Ay_0^4s/2) > \frac{1}{16}s^2y_0^4(B(2+Asy_0^3)^4 + 2A^2y_0^2(10+6Asy_0^3+A^2s^2y_0^6)) > 0,$$

this implies that $v_+(s) < y_0 + \epsilon = v(0) + v'(0)s + (5A^2v(0)^7/4 + Bv(0)^5)s^2$ for $|s| < \min((6|A|v(0)^3)^{-1}, (50\sqrt{B}v(0)^2)^{-1})$.

By an identical argument, for $\epsilon = Ay_0^4s/2 - (5A^2y_0^7/4 + By_0^5)s^2$ and $|s| = (HAy_0^3)^{-1}$ where $|H| > 6$,

$$\begin{aligned} Q_{-,s}(y_0+\epsilon) > \frac{1}{256A^{10}H^{10}y_0^{10}}(&-256B^5 - 256A^2B^4(11H^2)y_0^2 - 32A^4B^3(347H^4)y_0^4 \\ &- 16A^6B^2(1167H^6)y_0^6 + A^8B(86H^8)y_0^8), \end{aligned}$$

and setting $H := hy_0^{-1}A^{-1}\sqrt{B}$ where $h > 15$,

$$Q_{-,s}(y_0+\epsilon) > \frac{-128 - 1408h^2 - 5552h^4 - 9336h^6 + 43h^8}{128h^{10}} > 0.$$

For $As > 0$ and $|s| < \min(|318A|/(625By_0), (6|A|y_0^3)^{-1})$,

$$\begin{aligned} Q_{-,s}(y_0 + Ay_0^4 s) &= s(-Ay_0^3(2 + Asy_0^3) + Ay_0^3(1 + Asy_0^3)^3 - Bsy_0^4(1 + Asy_0^3)^4) \\ &< -\frac{s(318A + 625By_0 s)y_0^3}{1296} < 0. \end{aligned}$$

Therefore $y_0 + Ay_0^4 s/2 - (5A^2 y_0^7/4 + By_0^5)s^2 < v_-(s) < y_0 + Ay_0^4 s$ for $As > 0$. For $As < 0$ and $|s| < |A|/(By_0)$,

$$Q_{-,s}(y_0) = sy_0^3(A - Bsy_0) < 0,$$

so $y_0 + Ay_0^4 s/2 - (5A^2 y_0^7/4 + By_0^5)s^2 < v_-(s) < y_0$. Thus the lower bound holds in both cases.

The last statement follows because

$$\begin{aligned} |v(s) - v(0)| &< |v'(0)| \cdot |s| + \left(\frac{5}{144}R'''(t_{\max})^2 v(0)^7 + Bv(0)^5\right) s^2 \\ &< \frac{|R'''(t_{\max})|}{12} v(0)^4 |s| + \frac{5}{144} \frac{R'''(t_{\max})^2 v(0)^7}{|R'''(t_{\max})| v(0)^3} |s| + Bv(0)^5 \frac{53|R'''(t_{\max})|}{625 B v(0)} |s| \\ &< \frac{|R'''(t_{\max})| v(0)^4}{4} |s| < \frac{v(0)}{4}. \end{aligned}$$

(c) Differentiating (44) yields

$$(v(s) + sv'(s)) = \frac{-2s}{R'(t_{\max} + sv(s))}.$$

or equivalently

$$v'(s) = \frac{-2s - v(s)R'(t_{\max} + sv(s))}{sR'(t_{\max} + sv(s))}. \tag{48}$$

To estimate $v'(s)$, let us first estimate R'.

By a Taylor expansion,

$$|R'(t_{\max} + sv(s)) - R''(t_{\max})sv(s) - \frac{1}{2}R'''(t_{\max})s^2 v(s)^2| \leq 4Bs^3 v(s)^3. \tag{49}$$

By the triangle inequality and part (b),

$$\begin{aligned} &\left| R'(t_{\max} + sv(s)) - R''(t_{\max})s(v(0) + sv'(0)) - \frac{1}{2}R'''(t_{\max})s^2 v(0)^2 \right| \\ &\leq 4Bv(s)^3 s^3 - R''(t_{\max})((v(s) - v(0) - v'(0)s)s + \frac{1}{2}(v(s) - v(0))(v(s) + v(0))s^2 \\ &\qquad \leq \frac{125}{16}Bv(0)^3 s^3 - R''(t_{\max})\left(\frac{5}{144}R'''(t_{\max})^2 v(0)^7 + Bv(0)^5\right) s^3 \\ &\qquad\qquad + \frac{1}{2} \cdot \frac{1}{4} R'''(t_{\max})^2 v(0)^4 \cdot \frac{9}{4} v(0) s^3, \end{aligned}$$

which, by (45) and (46), implies

$$\left|R'(t_{\max}+sv(s))+2v(0)^{-1}s-4v(0)^{-2}v'(0)s^2\right| \\ \le \left(\frac{157}{16}Bv(0)^3+\frac{101}{288}R'''(t_{\max})^2v(0)^5\right)s^3 =: a_3s^3. \quad (50)$$

To estimate the inverse of R', use

$$\left|\frac{1}{a_1s+a_2s^2+a_3s^3}-\frac{1}{a_1s}+\frac{a_2}{a_1^2}\right| = \left|\frac{(a_2^2-a_1a_3)s^2+a_2a_3s^3}{a_1^2(a_1s+a_2s^2+a_3s^3)}\right| \le \left|\frac{6(a_2^2+|a_1a_3|)}{a_1^3}s\right| \\ \text{for } \ |s|<\min(\left|\frac{a_1}{3a_2}\right|,\left|\sqrt{\frac{a_1}{3a_3}}\right|),$$

which, by setting $a_1=2v(0)^{-1}$ and $a_2=4v(0)^{-2}v'(0)$, implies that

$$\left|\frac{1}{R'(t_{\max}+sv(s))}+\frac{v(0)}{2s}+v'(0)\right| \le \frac{R'''(t_{\max})^2v(0)^7+18v(0)^2|a_3|}{12}|s|. \quad (51)$$

Multiplying by $2|s|$ finishes the proof of (c).

(d) Differentiating (44) twice yields

$$2v'(s)+sv''(s)=\frac{-2-R''(t_{\max}+sv(s))(v(s)+sv'(s))^2}{R'(t_{\max}+sv(s))}. \quad (52)$$

From part (c) and a Taylor approximation for R'',

$$\left|-2-R''(t_{\max}+sv(s))(v(s)+sv'(s))^2\right| < 999((R'''(t_{\max})^2v(0)^6+Bv(0)^4)s^2 \\ +(R'''(t_{\max})^3v(0)^9+R'''(t_{\max})Bv(0)^7)s^3 \\ +(R'''(t_{\max})^4v(0)^{12}+R'''(t_{\max})^2Bv(0)^{10}+10B^2v(0)^8)s^4 \\ +(R'''(t_{\max})^5v(0)^{15}+R'''(t_{\max})^3Bv(0)^{13}+10R'''(t_{\max})B^2v(0)^{11})s^5 \\ +(R'''(t_{\max})^6v(0)^{18}+R'''(t_{\max})^4Bv(0)^{16}+10R'''(t_{\max})^2B^2v(0)^{14}+10B^3v(0)^{12})s^6).$$

Since $|s|<(R'''(t_{\max})v(0)^3)^{-1}, R'''(t_{\max})(Bv(0))^{-1}$,

$$\left|-2-R''(t_{\max}+sv(s))(v(s)+sv'(s))^2\right| < 99999(R'''(t_{\max})^2v(0)^6+Bv(0)^4)s^2. \quad (53)$$

By (51) and the estimates on $|s|$,

$$\left|\frac{1}{R'(t_{\max}+sv(s))}+\frac{v(0)}{2s}+v'(0)\right| \le 30R'''(t_{\max})v(0)^4. \quad (54)$$

Combining (52),(53) and (54),

$$|2v'(s)+sv''(s)-2v'(0)| < 50000(R'''(t_{\max})^2v(0)^7+Bv(0)^5)s \\ +400000(R'''(t_{\max})^3v(0)^{10}+R'''(t_{\max})Bv(0)^8)s^2,$$

and using $|s|<(R'''(t_{\max})v(0)^3)^{-1}$ on the second term gives the result.

□

Corollary 4.10. *Suppose that R and ϕ are infinitely continuously differentiable in some neighbourhood of t_{max}. Also suppose that t_{max} is a local maximum of R and $R''(t_{max}) < 0$. Let δ_1 and δ_2 be positive numbers such that*

$$m^2 := -R(t_{max} - \delta_1) = -R(t_{max} + \delta_2),$$

and assume m^2 equals the right-hand side of (47). *Let*

$$\tilde{s} = \min\left(\frac{R'''(t_{max})}{50Bv(0)}, \frac{1}{50R'''(t_{max})v(0)^3}\right),$$

$$\Lambda := 500R'''(t_{max})\|\phi'\|_\infty v(0)^5 + 450000\|\phi\|_\infty (R'''(t_{max})^2 v(0)^7 + Bv(0)^5).$$

Then for any $N > 1$,

$$\left|\int_{t_{max}-\delta_1}^{t_{max}+\delta_2} e^{NR(t)}\phi(t)dt - \phi(t_{max})\sqrt{\frac{-2\pi}{NR''(t_{max})}}\right| \leq$$

$$\frac{\sqrt{\pi}}{2}\frac{\Lambda}{N^{3/2}} + \phi(t_{max})\sqrt{\frac{-2}{R''(t_{max})}}\frac{e^{-N\tilde{s}}}{\sqrt{\tilde{s}}N} + e^{-N\tilde{s}^2/2}\Lambda.$$

Proof. Use Lemma 4.8. By part (a) of Lemma 4.9,

$$h(0) = \phi(t_{\max})\sqrt{\frac{-2}{R''(t_{\max})}}.$$

By parts (c) and (d) of Lemma 4.9,

$$\sup_{0 \leq \tau \leq m^2} |g'(\tau)| \leq \Lambda.$$

When $v(0) > \sqrt{B}/R'''(t_{\max})$,

$$\frac{53}{625}\frac{R'''(t_{\max})}{Bv(0)} > \frac{53}{625R'''(t_{\max})v(0)^3}$$

$$\frac{1}{50\sqrt{B}v(0)^2} > \frac{1}{50R'''(t_{\max})v(0)^3}$$

$$a_3 < 16R'''(t_{\max})^2 v(0)^5$$

$$\sqrt{\frac{2v(0)^{-1}}{3a_3}} > \frac{1}{5R'''(t_{\max})v(0)^3}.$$

When $v(0) < \sqrt{B}/R'''(t_{\max})$,

$$\frac{1}{R'''(t_{\max})v(0)^3} > \frac{R'''(t_{\max})}{Bv(0)}$$

$$\frac{1}{50\sqrt{B}v(0)^2} > \frac{R'''(t_{\max})}{50Bv(0)}$$

$$a_3 < 16Bv(0)^3$$

$$\sqrt{\frac{2v(0)^{-1}}{3a_3}} > \frac{1}{5\sqrt{B}v(0)^2} > \frac{R'''(t_{\max})}{5Bv(0)}.$$

Thus $m^2 > \tilde{s}$, and

$$\int_{m^2}^{\infty} e^{-Nt}t^{-1/2}dt \le \frac{e^{-Nm^2}}{mN} \le \frac{e^{-N\tilde{s}}}{\sqrt{\tilde{s}}N}.$$

□

We can finally wrap up the proof of Proposition 4.5. Since s is not too small (polynomial in $v(0)^{-1}$), the exponential terms are small enough to be ignored. Therefore the error term is $\Lambda_1 = \mathcal{O}(v(0)^7 N^{-3/2}) = \mathcal{O}(R_1''(t_{\max})^{-7/2}N^{-3/2})$. The main term is of order $N^{-1/2}R_1''(t_{\max})^{-1/2}$. Thus, when multiplying two integrals of the form in Corollary 4.10, we get

$$\mathcal{O}\left(\frac{R_1''(t_{\max_1})^{-3} + R_2''(t_{\max_2})^{-3}}{R_1''(t_{\max_1})^{1/2}R_2''(t_{\max_2})^{1/2}}N^{-2}\right) + \mathcal{O}(R_1''(t_{\max_1})^{-7/2}R_2''(t_{\max_2})^{-7/2}N^{-3}),$$

as needed. □

In Proposition 4.5, the error term blows up at the edge. Therefore a better bound is needed. To get this bound, we simply use the first term in Watson's lemma, as opposed to using two terms. Since the method of the proof is identical as before and the details are simpler, the proof will be omitted. The exact statement is the following.

Proposition 4.11. *For $j = 1, 2$, let $(\nu_j, \eta_j, \tau) \in \mathcal{D}$, Ω_j denote $\Omega(\nu_j, \eta_j, \tau)$, $G_j(z)$ denote $G(\nu_j, \eta_j, \tau, z)$, and θ_j denote $\theta(\nu_j, \eta_j, \tau)$. With the assumptions in section 2.1,*

$$\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{\exp(NG(\eta_1, \nu_1, \tau, u))}{\exp(NG(\eta_2, \nu_2, \tau, w))} f(u, w)dwdu$$
$$\le \frac{1000}{N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}} \times \left[\, |f(\Omega_1, \Omega_2)| + \left|f(\Omega_1, \bar{\Omega}_2)\right| + \left|f(\bar{\Omega}_1, \Omega_2)\right| + \left|f(\bar{\Omega}_1, \bar{\Omega}_2)\right| \,\right]$$

A Gaussian Free Field

This section is adapted from [11].

Given a domain $D \subset \mathbb{R}^d$, let $H_s(D)$ be the space of smooth, real-valued functions on $\mathbb{R}^d$ that are supported on a compact subset of D. Let $(f, g)_\nabla = \int_D (\nabla f \cdot \nabla g)dx$ be the Dirichlet inner product on $H_s(D)$. Let $H(D)$ be the

completion of $H_s(D)$ under the Dirichlet inner product. Note that $H(D)$ is a Hilbert space.

A **Gaussian free field** is any Gaussian Hilbert space $\mathcal{G}(D)$ of random variables denoted by $(h, f)_\nabla$ – one random variable for each $f \in H(D)$ – such that

$$\mathbb{E}[(h, a)_\nabla (h, b)_\nabla] = (a, b)_\nabla.$$

In other words, the map from $H(D)$ to $\mathcal{G}(D)$ defined by sending f to the random variable $(h, f)_\nabla$ preserves the inner product.

By the identity $(a, b) = \frac{1}{2}[(a+b, a+b) - (a, a) - (b, b)]$, this map preserves the inner product if it preserves the norm. In other words, the variance of $(h, f)_\nabla$ is $(f, f)_\nabla$ for each $f \in H(D)$.

At the most fundamental level, a Gaussian free field is a Gaussian Hilbert space. It turns out to be equivalent to consider other Hilbert spaces, which we will consider below.

Let $\{e_j\}$ be the eigenfunctions of the Dirichlet Laplacian on D which form an orthonormal basis for $L^2(D)$, and let $\{\lambda_j\}$ denote the corresponding eigenvalues. By the spectral theorem for compact self-adjoint operators, all the λ_j are negative. Suppose the ordering satisfies $\lambda_1 \geq \lambda_2 \geq \ldots$. Define the operator on $L^2(D)$:

$$(-\Delta)^a \sum \beta_j e_j = \sum (-\lambda_j)^a \beta_j e_j.$$

Define $\mathcal{L}_a(D) := (-\Delta)^a L^2(D)$ to be the set of all $\sum \beta_j e_j$ such that $\sum (-\lambda_j)^{-a} \beta_j e_j \in L^2(D)$. We may view $\mathcal{L}_a(D)$ has a Hilbert space with inner product $(f, g)_a = ((-\Delta)^{-a} f, (-\Delta)^{-a} g)$. Note that when $a < 0$, the sum $\sum \beta_j e_j$ may not converge in $L^2(D)$, but it converges in the space of distributions.

Note that by integration by parts, $H(D) = \mathcal{L}_{-1/2}(D)$. Therefore, instead of considering $H(D)$ as the Hilbert space, we can also consider $\mathcal{L}_{1/2} = (-\Delta) H(D)$. Then

$$\mathbb{E}[(h, f)_{1/2} (h, g)_{1/2}] = (f, g)_{1/2} = ((-\Delta)^{-1/2} f, (-\Delta)^{-1/2} g) = ((-\Delta)^{-1} f, g).$$

Since

$$[(-\Delta)^{-1} f](x) = \int_D G(x, y) f(y) dy,$$

this implies

$$\mathrm{Cov}[(h, f)_{1/2}, (h, g)_{1/2}] = \int_{D \times D} f(y) g(x) G(x, y) dx dy.$$

The higher moments can be computed from the following theorem,

Theorem A.1. *If $X_1, \ldots, X_k$ are mean zero random variables such that $a_1 X_1 + \ldots + a_k X_k$ is Gaussian for all $a_1, \ldots, a_k \in \mathbb{R}$, then*

$$\mathbb{E}[X_1 \ldots X_k] = \begin{cases} \displaystyle\sum_\sigma \prod_{j=1}^{k/2} \mathbb{E}[X_{\sigma(j)} X_{\sigma(j+1)}], & k \text{ even} \\ 0, & k \text{ odd}, \end{cases}$$

where the sum is over all involutions on $\{1, \ldots, k\}$ with no fixed point.

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